14.102 Problem Set 2 Solutions

1. Lecture Notes Exercise 78: Consider the 2×2 identity matrix. What are its eigenvalues? Find a $V = [v_1 \ v_2]$ such that $V'V = I$ and $V^{-1}IV = I$. What are the corresponding $\{\mathbb{M}_1, \mathbb{M}_2\}$?

   Solution: The 2×2 identity matrix has one eigenvalue, 1, with multiplicity two. Any vector belonging to $\mathbb{R}^2$ will work as an eigenvector, so we can freely choose two that satisfy the conditions given. The natural choice is $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that this makes V be the 2×2 identity matrix itself, which is its own transpose and inverse - so we have nothing but identity matrices in the equations $V'V = I$ and $V^{-1}IV = I$. $\mathbb{M}_1 = \mathbb{M}_2 = \mathbb{R}^2$, because each characteristic manifold is the span of all eigenvectors associated with the eigenvalue 1 - which, again, are all vectors in $\mathbb{R}^2$.

   Consider now

   $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}$$

   Find an orthonormal V and a diagonal Λ such that $V'AV = \Lambda$. Hint: remember that if v is an eigenvector, then ϕv is also an eigenvector for any scalar ϕ.

   Solution: Expanding along the first row or first column, we have

   $$|A - \lambda I| = (1 - \lambda)[(2 - \lambda)(1 - \lambda) - 9] = 0$$

   The roots of this equation are 1, $\frac{3 \pm \sqrt{37}}{2}$. Note that these eigenvalues are distinct. Recall that we spent some time in class going over the correction of a theorem which said that eigenvectors for distinct eigenvalues are orthogonal. The correction (which is also captured in a separate handout on the course website) pointed out that this was true only for symmetric matrices. Since A is symmetric, we know this will be the case; if we find any three eigenvectors, they will necessarily be orthogonal to one another (we also know that eigenvalues and eigenvectors are real). So first we'll just find an eigenvector for each eigenvalue.

   (a) $\lambda_1 = 1:$ We need $|A - I|\, v_1 = 0$. That is,

   $$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 0 \end{bmatrix} v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

   It is clear that any vector with zeroes in its second two components will work, so we will set $v_1 = (1, 0, 0)$. Notice that this choice implies that we will have to set the first component of the other two eigenvectors to zero to have orthogonality.

(b) $\lambda_2 = \frac{3+\sqrt{37}}{2}$: We have

$$\begin{bmatrix} -\frac{1}{2}-\frac{\sqrt{37}}{2} & 0 & 0 \\ 0 & \frac{1}{2}-\frac{\sqrt{37}}{2} & 3 \\ 0 & 3 & -\frac{1}{2}-\frac{\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This gives us the system

$$\begin{aligned} (\frac{1}{2}-\frac{\sqrt{37}}{2})\alpha + 3\beta &= 0 \\ 3\alpha + (-\frac{1}{2}-\frac{\sqrt{37}}{2})\beta &= 0 \end{aligned}$$

Which tells us that $\beta = (\frac{\sqrt{37}-1}{6})\alpha$; any vector with α and β in this ratio will work. The natural choice is $v_2 = (0, 1, \frac{\sqrt{37}-1}{6})$.

(c) $\lambda_3 = \frac{3-\sqrt{37}}{2}$: Following the same procedure,

$$\begin{bmatrix} -\frac{1}{2}+\frac{\sqrt{37}}{2} & 0 & 0 \\ 0 & \frac{1}{2}+\frac{\sqrt{37}}{2} & 3 \\ 0 & 3 & -\frac{1}{2}+\frac{\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} 0 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

yields

$$\begin{aligned} (\frac{1}{2}+\frac{\sqrt{37}}{2})\alpha + 3\beta &= 0 \\ 3\alpha + (-\frac{1}{2}+\frac{\sqrt{37}}{2})\beta &= 0 \end{aligned}$$

or $\beta = -(\frac{\sqrt{37}+1}{6})\alpha$. We can set $v_3 = (0, 1, -\frac{\sqrt{37}+1}{6})$.

These three eigenvectors are orthogonal, but not orthonormal because $v_2'v_2 \neq 1$ and $v_3'v_3 \neq 1$. But recall that any scalar multiple of an eigenvector is still an eigenvector - so we can normalize these three as needed.

v_1 is fine. To normalize v_2, observe that $v_2'v_2 = 0+1+\frac{1}{36}(37-2\sqrt{37}+1) = \frac{1}{18}(37-\sqrt{37})$. If we call this number ϕ^{-2}, then $(\phi v_2)'(\phi v_2) = \phi^2 v_2'v_2 = 1$. Similarly, $v_3'v_3 = \frac{1}{18}(37+\sqrt{37}) \equiv \psi^{-2}$; then $(\psi v_3)'(\psi v_3) = \psi^2 v_3'v_3 = 1$. So our orthonormal V is $(v_1, \phi v_2, \psi v_3)$, and $\Lambda = diag(\lambda_1, \lambda_2, \lambda_3)$, the diagonal matrix with the three eigenvalues on the diagonal.

2. Lecture Notes Exercise 91: Show that if X is symmetric and idempotent, then X is also positive semi-definite. Note that prior to 9/29, 'and idempotent' was missing from the lecture notes, but is needed! Optional: can you see why?

   Solution: $X = XX = X'X$. So define $y = Xa$, $a \in \mathbb{R}^n$. Then $a'Xa = a'X'Xa = y'y = \sum_i y_i^2 \geq 0$, so X is positive semi-definite.

3. Give an example of a function that is continuous at exactly one point (say, 0) and is also differentiable at this point.

   Solution: As in class, define function $D(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$. Then $f(x) = x^2 D(x)$ will work.

4. Lecture Notes Exercise 122: Find the domains of the following functions $f : \mathbb{R} \to \mathbb{R}$:

   (a) $f(x) = \sqrt{x}$ (note: we typically adopt the convention that the square root of x refers to the positive root, unless explicitly stated otherwise.
   Solution: $\{x \in \mathbb{R} : x \geq 0\}$

   (b) $f(x) = \frac{1}{x^2+2x-3}$
   Solution: $\{x \in \mathbb{R} : x^2 + 2x - 3 \neq 0\} = \{x \in \mathbb{R} : x \neq -1, x \neq 3\}$

   (c) $f(x) = \frac{1}{\sin x} + \frac{1}{\cos x}$
   Solution: $\{x \in \mathbb{R} : \sin x \neq 0, \cos x \neq 0\} = \{x \in \mathbb{R} : x \neq \frac{n\pi}{2}, n \in \mathbb{Z}\}$

5. (Sundaram 4.4, page 110) For each of the following functions, state whether the conditions of the Weierstraß theorem apply. Find and classify all critical points (local maximum, local minimum, neither) of each of the following functions. For local optima that you find figure out whether they are also global optima. Try to save your time by avoiding using second order approach wherever possible.

   Solution: For none of these functions is the domain compact, so the conditions of Weierstraß do not apply. However, as these conditions are only sufficient for extrema, this does not imply that we cannot find local or global optima.

   (a) $f(x, y) = x \sin y$
   Solution: At any critical point we have $\sin y = 0$, meaning that $\cos y = \pm 1 \neq 0$ and hence $x = 0$. So critical points are $(0, \pi k)$, $k \in \mathbb{Z}$.Given that f is periodic in y, all these points look "the same" as $(0, 0)$ which is clearly neither maximum nor minimum ($f(x, y) \approx xy$ around $(0, 0)$, so f can go both slightly above and slightly below 0 for small x and y). So no local extrema.

   (b) $f(x, y) = \frac{1}{x} + \frac{1}{y}$
   Solution: Here the failure to satisfy Weierstrass does in fact mean that we cannot find maxima or minima on the domain of this function.
   Note: suppose we introduce a constraint that makes the domain compact - then we should be able to find extrema. For example - $f(x, y) = \frac{1}{x} + \frac{1}{y}$ subject to $\left(\frac{1}{x}\right)^2 + \left(\frac{1}{y}\right)^2 = \left(\frac{1}{a}\right)^2$. We may change variables to $u = \frac{1}{x}$ and $v = \frac{1}{y}$ and then maximize and minimize

$u+v$ on the circle $u^2+v^2=\left(\frac{1}{a}\right)^2$, with four points omitted (those where either u or v is zero) that makes the problem not compact and hence Weierstraß theorem not applicable. Geometrically it is clear that maximum will be at $u=v=\frac{1}{a\sqrt{2}}$ and the minimum at $u=v=-\frac{1}{a\sqrt{2}}$ (respectively, $x=y=a\sqrt{2}$ and $x=y=-a\sqrt{2}$).

(c) $f(x,y)=x^4+y^4-x^3$

Solution: A function $f(x,y)$ that is decomposable as $f(x,y)=g(x)+h(y)$ has critical points at points (x_0,y_0) if and only if x_0 is a critical point for g and y_0 is a critical point for h (why?). Moreover, (x_0,y_0) is a local minimum (maximum) of f if and only if x_0 and y_0 are local minima (maxima) of g and h simultaneously.

Now, $h(y)=y^4$ has only one critical point $y_0=0$ and it is a local minimum. Consequently, $f(x,y)$ does not have local maxima at all and only has a local minimum wherever $g(x)=x^4-x^3$ has a local minimum, which apparently is at point $x_0=\frac{3}{4}$. The other critical point of f is $(0,0)$ which is local minimum for h but not an extremum for g, so is not an extremum for f either.

Point $(\frac{3}{4},0)$ is also the global minimum for $f(x,y)$ (which is bounded from below).

6. (Simon and Blume 15.6, page 342) Consider the function $F(x_1,x_2,y)=x_1^2-x_2^2+y^3$.

(a) If $x_1=6$ and $x_2=3$, find a y which satisfies $F(x_1,x_2,y)=0$.

Solution: We have $36-9+y^3=0$, or $y=-3$.

(b) Does this equation define y as an implicit function of x_1 and x_2 near $x_1=6$, $x_2=3$?

Solution: Given $(x_1^*,x_2^*,y^*)=(6,3,-3)$, we have $\frac{\partial F}{\partial y}(x_1^*,x_2^*,y^*)=3y^{*2}=27\neq 0$, so it does.

(c) If so, compute $(\frac{\partial y}{\partial x_1})(6,3)$ and $(\frac{\partial y}{\partial x_2})(6,3)$.

Solution: $\frac{\partial y}{\partial x_1}(6,3)=-\frac{\frac{\partial F}{\partial x_1}}{\frac{\partial F}{\partial y}}(6,3,y^*(6,3))=-\frac{2x_1}{3y^2}(6,3,y^*(6,3))=-\frac{12}{27}$

$\frac{\partial y}{\partial x_2}(6,3)=-\frac{\frac{\partial F}{\partial x_2}}{\frac{\partial F}{\partial y}}(6,3,y^*(6,3))=\frac{2x_2}{3y^2}(6,3,y^*(6,3))=\frac{6}{27}$

(d) If x_1 increases to 6.2 and x_2 decreses to 2.9, estimate the corresponding change to y.

Solution: $\frac{\partial y}{\partial x_1}(6,3)*(0.2)+\frac{\partial y}{\partial x_2}(6,3)*(-0.1)=(-\frac{12}{27})*(0.2)+(\frac{6}{27})*(-0.1)=-\frac{3}{27}$, so y changes by $-\frac{1}{9}$.