14.102 Problem Set 3

Due Thursday, October 21, 2004, in class

Starred (*) problems will not count for the grade on this problem set; they are based on material from lectures on 10/21 and 10/26, and provide practice for the midterm on 10/28. If you would like to do them prior to 10/21 and hand them, I will be happy to make comments. Solutions to all problems will be posted after class on 10/21.

1. Lecture Notes Exercise 168: For a function f defined on a convex subset U in $\mathbb{R}^n$, show that f concave implies f quasiconcave.

   Solution: Such an f satisfies $f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y)$ for any $x, y \in U$ and $0 < \lambda < 1$. We will show that this means that $f(x) \geq f(y)$ implies $f(\lambda x + (1-\lambda)y) \geq f(y)$, again for any $x, y \in U$ and $0 < \lambda < 1$. So suppose that we have chosen x and y such that $f(x) \geq f(y)$. Then we have

   $$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \geq \lambda f(y) + (1-\lambda)f(y) = f(y)$$

2. (Sundaram 5.11, page 144) Consider the problem of maximizing the utility function $u(x, y) = x^{\frac{1}{2}} + y^{\frac{1}{2}}$ on the budget set $px + y = 1, x \geq 0, y \geq 0$. Show that if non-negativity constrains are ignored, and the problem is written as an equality-constrained one, the resulting Lagrangean has a unique critical point. Does this critical point identify a solution to the problem? Why or why not?

   Solution: We have $L(x, y, \lambda) = \sqrt{x} + \sqrt{y} - \lambda(px + y - 1)$. The only critical point is given by $\frac{1}{2\sqrt{x}} = p\lambda$ and $\frac{1}{2\sqrt{y}} = \lambda$, so $y = p^2 x$. Plugging that to the budget constraint gives $x = \frac{1}{p+p^2}$, $y = \frac{p}{1+p}$.

   This unique critical point is indeed the solution to our problem: generally, if we ignore some constraints and find a solution to the problem with relaxed constraints, and it so happens that those relaxed constraints are satisfied at the solution we find, then this is also a solution to the problem with relaxed constraints. Make sure you understand this logic.

3. (Sundaram 6.12, page 171) A firm produces a single output y using three inputs x_1, x_2, x_3 in nonnegative quantities through the relationship $y = x_1(x_2 + x_3)$.The unit price of y is $p_y > 0$ while that of the input x_i is $w_i > 0, i = 1, 2, 3$.

   (a) Describe the firm's profit-maximization problem and derive the equations that define the critical points of the Lagrangean L in this problem.

   Solution: The firm want to maximize $p_y y - w_1 x_1 - w_2 x_2 - w_3 x_3$ subject to $y = x_1(x_2 + x_3)$ and $x_1, x_2, x_3 \geq 0$. We may immediately plug the first constraint to the objective to make life easier.

The Lagrangian is $L(x_1, x_2, x_3, \lambda_1, \lambda_2, \lambda_3) = p_y x_1(x_2 + x_3) - w_1 x_1 - w_2 x_2 - w_3 x_3 + \lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3$. Critical points are then given by:

$$\begin{cases} p_y(x_2 + x_3) &= w_1 - \lambda_1 \\ p_y x_1 &= w_2 - \lambda_2 \\ p_y x_1 &= w_3 - \lambda_3 \\ \lambda_1 x_1 &= 0 \\ \lambda_2 x_2 &= 0 \\ \lambda_3 x_3 &= 0 \end{cases}$$

(b) Show that the Lagrangean L has multiple critical points for any choice of $(p_y, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$.

Solution: $x_1 = x_2 = x_3 = 0$, $\lambda_i = w_i$, $i = 1, 2, 3$ is always a critical point. So is $\lambda_1 = \lambda_2 = 0 \neq \lambda_3$ for $w_3 > w_2$ and $\lambda_1 = \lambda_3 = 0 \neq \lambda_2$ for $w_2 > w_3$. For $w_2 = w_3$ any x_2 and x_3 such that $x_2 + x_3 = \frac{w_1}{p_y}$ is a critical point.

(c) Show that none of these critical points identifies a solution of the profit-maximization problem. Can you explain why this is the case?

Solution: This is the case simply because there is no solution to the profit-maximization problem at all. That is, the profits can be potentially made infinite. To see this, consider moving along the line $x_1 = x_2 = a$, $x_3 = 0$. Profits then are $p_y a^2 - (w_1 + w_2)a$, which grows to infinity as $a \to \infty$.

4. (Sundaram 8.25, page 201) An agent who consumes three commodities has a utility function given by $u(x_1, x_2, x_3) = \sqrt[3]{x_1} + \min\{x_2, x_3\}$. Given an income of I and prices p_1, p_2, p_3, write down the consumer's utility-maximization problem (you need not solve it[1]). Can the Weierstraß and/or Kuhn-Tucker theorems be used to obtain and characterize a solution (that is, are they applicable to this problem)? Why or why not?

   Solution: The problem is to maximize $u(x_1, x_2, x_3) = \sqrt[3]{x_1} + \min\{x_2, x_3\}$ subject to $p_1 x_1 + p_2 x_2 + p_3 x_3 \leq I$. The Weierstraß theorem is applicable, as long as prices are strictly positive, but Kuhn-Tucker theorem is not applicable, since u is not a C^1 function (it is not differentiable at points where $x_2 = x_3$). The way to solve it will be to notice that it is never optimal to have $x_2 \neq x_3$ (if, say, $x_2 < x_3$, then cutting down on x_3 and buying some more of x_2 will improve utility). Therefore, we may denote by x the composite good composed of equal quantities of x_2 and x_3, which is going at price $p = p_2 + p_3$. The problem then becomes to maximize $u(x_1, x) = \sqrt[3]{x_1} + x$ subject to $p_1 x_1 + px \leq I$; now Kuhn-Tucker theorem is applicable.

5. Lecture Notes Exercise 205: Compute $\int_a^{+\infty} te^{-rt} dt$ (use integration by parts).

[1] If you ever take a class from Bengt, or an exam he writes, you will become very used to problems that require you only to set up the program (but which can still be surprisingly difficult!).

Solution: $\int_a^{+\infty} te^{-rt}dt = [-\frac{t}{r}e^{-rt}]_a^{+\infty} + \int_a^{+\infty} \frac{1}{r}e^{-rt}dt = [-\frac{t}{r}e^{-rt} - \frac{1}{r^2}e^{-rt}]_a^{+\infty} = \frac{ra-1}{r^2}e^{-ra}$

6. Lecture Notes Exercise 206: Compute $\int_0^{+\infty} e^{-\sqrt{t}}dt$ (use the change of variable $u = \sqrt{t}$).

   Solution: $u = \sqrt{t}$, so $dt = 2udu$. The limits of integration don't change, so we have
   $\int_0^{+\infty} e^{-\sqrt{t}}dt = \int_0^{+\infty} 2ue^{-u}du = 2\{[-ue^{-u}]_0^{+\infty} + \int_0^{+\infty} e^{-u}du\} = 2[-e^{-u}(u+1)]_0^{+\infty} = 2$

7. Lecture Notes Exercise 211: For each of the following relations, show that R is (or is not) reflexive, symmetric, and transitive. In each, $x, y \in \mathbb{R}^n$.

   (a) xRy if $x_1 > y_1$, where x_1 and y_1 are the respective first elements of x and y.

   Solution: $x_1 > x_1$ cannot hold, so R is not reflexive. Moreover, only one of $\{x_1 > y_1, x_1 < y_1, x_1 = y_1\}$ can hold, so R is not symmetric. But if $x_1 > y_1$, and $y_1 > z_1$ for a third vector z, then $x_1 > z_1$; R is transitive.

   (b) xRy if $x_1 = y_1$

   Solution: $x_1 = x_1$ must hold; $x_1 = y_1 \Longrightarrow y_1 = x_1$; and if $x_1 = y_1$ and $y_1 = z_1$, then $x_1 = z_1$; thus, R is reflexive, symmetric, and transitive (it is an equivalence relation - note the implication that an equivalence relation need not be *equality).*

   (c) xRy if $\|x\| = \|y\|$

   Solution: We have exactly the same arguments as in part (b) leading to the conclusion that R is an equivalence relation (reflexive, symmetric, and transitive). Note that $\|\cdot\|$ is a function from $\mathbb{R}^n$ to $\mathbb{R}$. We cannot put an order on n-tuples (vectors); it may not always be possible to say $x > y$, $x < y$, or $x = y$ for *any* pair $x, y \in \mathbb{R}^n$. But a function mapping vectors onto the real line can help us to do this, because we can order the real numbers. This is one aspect of the utility function - it takes bundles of goods which are not inherently comparable and maps them onto the real line, allowing us to rank them according to a preference relation. Can you tell if $u(x) = \|x\|$ could be a utility function?

8. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all real numbers $-x$, where $x \in A$. Prove that

$$\inf A = -\sup(-A)$$

   Solution: First we note that because A is bounded below, $\inf A$ exists in $\mathbb{R}$. Call it α. Now, by definition, α is a lower bound for A, but if there exists a $\gamma > \alpha$, γ is not a lower bound for A. Now consider $-\alpha$.

Because $\alpha \leq x$ for all $x \in A$, we know that $-\alpha \geq -x$ for all $-x \in -A$. Thus, $-\alpha$ is an upper bound for $-A$ (Note: if A is bounded below, then we know that $-A$ is bounded above, and hence that $\sup(-A)$ exists in $\mathbb{R}$). Now consider $\beta < -\alpha$. If β is an upper bound for $-A$, then $\beta \geq y$ for all $y \in -A$. But this would imply, first, that $-y \in A$, and, second, that $-y \geq -\beta > \alpha$, i.e. that there exists a real number greater than α which is a lower bound for A. But we began by pointing out that this is not the case, since $\alpha = \inf(A)$. Thus, any such β is not an upper bound for $-A$, which means that $-\alpha = \sup(-A)$, which in turn implies that $\alpha = \inf A = -\sup(-A)$.

9. (*) Prove carefully that the sum of two convergent sequences is convergent and its limit is the sum of the limits.

   Solution: Let $\{a_n\} \to a$ and $\{b_n\} \to b$. We want to show that $\{a_n+b_n\} \to a+b$. By definition of convergence, this means that $\forall \varepsilon > 0\ \exists N :\ \forall n > N$ $|a_n + b_n - (a+b)| < \varepsilon$. Indeed, for a given $\varepsilon > 0$ consider $\varepsilon_1 = \frac{\varepsilon}{2}$. By definition of convergence, there must exist N_a such that $\forall n > N_a$ $|a_n - a| < \varepsilon_1$; likewise, there must exist N_b such that $\forall n > N_b$ $|b_n - b| < \varepsilon_1$. Set $N = \max\{N_a, N_b\}$. Now for any $n > N$ we have $|a_n + b_n - (a+b)| = |a_n - a + b_n - b| \leq |a_n - a| + |b_n - b| < \varepsilon_1 + \varepsilon_1 = \varepsilon$. This completes the proof.

10. (*) Find all limit points of the following sequence: 1, 1, 2, 1, 2, 3, 1, 2, 3, 4,...

    Solution: All positive integers are limit points of this sequence.

11. (*) Show that the intersection of (even infinitely many) closed sets is closed. Give an example of an infinite family of closed sets whose union is not closed.

    Solution: By definition, a set is closed if its complement is open. We have to prove that the complement of an intersection of closed sets is open, which is the same as the union of complements of each of them (why?), which are all open by definition. Therefore, we have to prove that the union of (any number of) open sets is open. The latter statement is straightforward: if a point belongs to the union, it belongs to one of the sets which, being open, contains a small enough open ball centered in our point and this ball must, therefore, be contained in the union, proving that the union is open.

    Make sure you understand the logic of the above solution and can, if needed, reproduce it.

    If $A_n = \{\frac{1}{n}\}$, then each A_n is closed (it is a singleton), but their union is not (limit point 0 is missing).

12. (*) Let $A = [-1; 0)$ and $B = (0, 1]$. Examine whether each of the following statements is true or false:

(a) $A \cup B$ is compact;

Solution: False. Point 0 is missing making the set not closed.

(b) $A + B = \{x + y | x \in A, y \in B\}$ is compact;

Solution: False. $A + B = (-1; 1)$ which is not closed.

(c) $A \cap B$ is compact.

Solution: True. $A \cap B = \emptyset$ which is closed and bounded.

13. (*) Define

$$f(x) = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Find an open set O such that $f^{-1}(O)$ is not open and a closed set C such that $f^{-1}(C)$ is not closed.

Examples: $O = (\frac{1}{2}, \frac{3}{2})$ and $C = \{0\}$

14. (*) (Harder) Start from any set $A \in \mathbb{R}^n$. Consider the following two operations: taking closure of a set and taking convex hull of a set. At most how many *distinct* sets can one obtain by consecutively applying these operations to A (in any order)? Try to show that the number you get is indeed the maximum.

Solution: Convince yourself that the closure of a convex set is convex (I will try to explain it in class). Now if you start from any set A, you can go to $cl(A)$ then to $con(clA)$ then to $cl(con(clA)$ at which point you end up with a closed convex set, so you can go no further. Therefore maximum number of distinct sets that you get can not go above four. This bound can indeed be achieved: if $A = \{(x, y) \in \mathbb{R}^2 | x \in \mathbb{Q}, x \neq 0, y = \frac{1}{|x|}\}$. Make sure you can see what $A, cl(A), con(cl(A))$ and $cl(con(cl(A)))$ are and that they are indeed all distinct.