14.102 Problem Set 4

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1 Optimization in Discrete Time

We're going to apply the tools we used in class using the Lagrange multiplier to the following problem with many more variables and with two constraints. Note that interest rate R is exogenous here. Since we only want to characterize laws of motion, we're going to ignore initial and terminal conditions.

$$\max_{\substack{\{c_t\}_{t=0}^{\infty} \\ \{k_{t+1}\}_{t=0}^{\infty} \\ \{a_{t+1}\}_{t=0}^{\infty} \\ \{i_t\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

s.t.

$$a_{t+1} = -c_t + F(k_t) - i_t\left(1 + \gamma \frac{i_t}{k_t}\right) + Ra_t, \text{ and}$$

$$k_{t+1} = (1-\zeta) k_t + i_t$$

1. Write out the problem in Lagrange form. Let λ_t represent the multiplier for the first constraint and μ_t be the multiplier for the second constraint.

2. Determine the first order conditions with respect to c_t, k_{t+1}, a_{t+1}, and i_t. Do second order conditions hold? Under what circumstances will they hold? From now on, assume all first order conditions hold with equality.

3. Define a new variable $q_t = \mu_t/\lambda_t$. Rewrite the first order condition with respect to i_t such that you have an equation for i_i/k_t as a function of q_t

4. Combine the first order condition for k_{t+1} with the first order condition for a_{t+1} in order to describe q_t as a function of q_{t+1}, k_{t+1}, i_{t+1}

5. Combine the answer in (4) with the answer in (3) to write q_t as a function of q_{t+1} and k_{t+1} (get rid of i_{t+1}).

6. Combine the answer in (3) with the law of motion of capital to solve for k_{t+1} as a function of k_t and q_t.

7. Now that you have two difference equations in q and k, What value of q_t set $k_t = k_{t+1}$ as required in steady state? What values of k_t set $q_t = q_{t+1}$ as required in steady state?

2 Optimization in Continuous Time

Let's solve the same problem in continuous time (Note: $r = R - 1$, although this is not important to you solving the problem)

$$\max_{c_t, k_t, a_t, i_t} \int_0^\infty e^{-\rho t} U(c_t)\, d_t \tag{2}$$

s.t.

$$\dot{a_t} = -c_t + F(k_t) - i_t \left(1 + \gamma \frac{i_t}{k_t}\right) + r a_t$$

$$\dot{k_t} = -\zeta k_t + i_t$$

1. Write out the problem in Lagrange form. Let λ_t represent the multiplier for the first constraint and μ_t be the multiplier for the second constraint. Normalize each multiplier by $e^{-\rho t}$.

2. Before converting the equation into a present value Hamiltonian make the following change of variable: let $q_t = \mu_t / \lambda_t$, and rewrite the Lagrange as a function of λ_t and q_t alone. Convert the equation into a present value Hamiltonian which omits $\dot{a_t}$ and $\dot{k_t}$ and includes $\dot{\lambda_t}$ and $\dot{q_t}$ instead.

3. Determine the first order conditions for c_t, k_t, a_t, and i_t.

4. Combine the first order condition for a_t and the first order condition for k_t to cancel out all terms with λ_t in the equation for k_t.

5. Plug the first order conditions for i_t into the equation just derived so that you are left for an expression of $\dot{q_t}$ as a function of q_t and k_t

6. Plug the first order condition for i_t into the law of motion on capital so that you have an expression of $\dot{k_t}$ as a function of k_t and q_t

7. Now that you have two differential equations in q and k, What value of q_t set $\dot{k}_t = 0$ as required in steady state? What values of k_t set $\dot{q}_t = 0$ as required in steady state?

3 Phase Diagrams

1. Using the result in part 7 of question 2, draw a phase diagram with q on the y axis and k on the x axis

2. Is the system globally stable? Is it locally stable? If it is locally stable, where is the stable arm?

3. Now, assume that you are in the steady state and there is an exogenous increase in γ. Describe the path of the economy, so describe in words what happens to k_t and to q_t. (HINT: when an exogenous shock occurs, we jump to the new stable arm immediately, and then follow it to the new steady state).

4 Infinite Sums

1. Take the equation for q_t as a function of q_{t+1}, k_{t+1}, AND i_t from problem 1. (make sure to keep the i_t term in there, otherwise you have a quadratic in q_{t+1}. Plug in recursively forward for q_{t+1} so that you have q_t being equal to an infinite sum. What does this sum look like (you do not have to solve for anything explicitly, just plug in and explain intuitively what it looks like)? It turns out that q_t refers to Tobin's q here, so perhaps you can see from the result that this term tells us something about the future productive capacities of the economy.

2. Take the equation of $\dot{q}_t$ as a function of q_t, k_t, and i_t from problem 2 (as before, do not plug in for i_t). Move all of the terms involving q to the same side (Hint: you will be left with $\dot{q}_t - \rho q_t$ on one side). Multiply both sides by $e^{-\rho t}$ and integrate the two sides from t to ∞. You should have an expression (you do not have to solve explicitly for it) for q_t. What does this sum look like?

3. Here is a different scenario. Imagine I have an exogenous income stream of $\overline{y}$ which I receive in every period so that $y_t = \overline{y}\ \forall t$. There are two interest rates in the economy. For periods $t \in [0, T_1]$ the interest rate is $1 + r_1$. For periods $t \in (T_1, \infty]$, the interest rate is $1 + r_2$.

(a) Assume we are in discrete time. Write out the present discounted value of my income stream from the standpoint of $t = 0$

(b) Assume we are in continuous time. Write out the present discounted value of my income stream from the standpoint of $t = 0$

(c) Assume the same setup in discrete time, but instead, we have an arbitrary interest rate sequence $\{r_t\}_{t=0}^{\infty}$. What is the present discounted value of my income stream now?

(d) Assume the same setup in continuous time, but instead we have an arbitrary interest rate function $r(t)$. What is the present discounted value of my income stream now?