14.102 Problem Set 4 Solutions

Fall, 2004
Lecture: Pierre Yared
TA: Nathan Barczi

1 Optimization in Discrete Time

We're going to apply the tools we used in class using the Lagrange multiplier to the following problem with many more variables and with two constraints. Note that interest rate R is exogenous here. Since we only want to characterize laws of motion, we're going to ignore initial and terminal conditions.

$$\max_{\substack{\{c_t\}_{t=0}^{\infty} \\ \{k_{t+1}\}_{t=0}^{\infty} \\ (a_{t+1}\}_{t=0}^{\infty} \\ \{i_t\}_{t=0}^{\infty}}} \sum_{t=0}^{\infty} \beta^t U(c_t) \tag{1}$$

$$s.t.$$

$$\begin{aligned} a_{t+1} &= -c_t + F(k_t) - i_t(1 + \gamma \frac{i_t}{k_t}) + Ra_t, \text{ and} \\ k_{t+1} &= (1 - \varsigma)k_t + i_t \end{aligned}$$

1. Write out the problem in Lagrange form. Let λ_t represent the multiplier for the first constraint and μ_t be the multiplier for the second constraint.

   Solution: $L(c_t, k_{t+1}, a_{t+1}, i_t, \lambda_t, \mu_t) = \sum_{t=0}^{\infty} \{\beta^t U(c_t) - \lambda_t[a_{t+1} + c_t - F(k_t) + i_t(1 + \gamma \frac{i_t}{k_t}) - Ra_t] - \mu_t[k_{t+1} - (1 - \varsigma)k_t - i_t]\}$

2. Determine the first order conditions with respect to c_t, k_{t+1}, a_{t+1}, and i_t. Do second order conditions hold? Under what circumstances will they hold? From now on, assume all first order conditions hold with equality.

   Solution:

$$\begin{aligned} c_t &: \beta^t U'(c_t) - \lambda_t \geq 0 \\ k_{t+1} &: \lambda_{t+1}[F'(k_{t+1}) + \gamma(\frac{i_{t+1}}{k_{t+1}})^2] - \mu_t + \mu_{t+1}(1 - \varsigma) \geq 0 \\ a_{t+1} &: \lambda_t - \lambda_{t+1} R \geq 0 \\ i_t &: -\lambda_t(1 + 2\gamma \frac{i_t}{k_t}) + \mu_t \geq 0 \end{aligned}$$

   The Hessian for this system is diagonal, so that checking second-order conditions consists of checking that each second derivative is negative. This is the case so long as $U''(c_t)$ and $F''(k_t)$ are negative, which we generally assume to be the case.

3. Define a new variable $q_t = \mu_t/\lambda_t$. Rewrite the first order condition with respect to i_t such that you have an equation for i_t/k_t as a function of q_t.

   Solution: Simply divide through by λ_t to find

$$\frac{i_t}{k_t} = \frac{q_t - 1}{2\gamma}$$

4. Combine the first order condition for k_{t+1} with the first order condition for a_{t+1} in order to describe q_t as a fucntion of $q_{t+1}, k_{t+1}, i_{t+1}$.

   Solution: Divide through the first order condition for k_{t+1} by λ_{t+1} to get

$$F'(k_{t+1}) + \gamma(\frac{i_{t+1}}{k_{t+1}})^2 - \frac{\mu_t}{\lambda_{t+1}} + \frac{\mu_{t+1}}{\lambda_{t+1}}(1-\varsigma) = 0$$

   and noting that, from the first order condition for a_{t+1}, $\lambda_{t+1} = \frac{\lambda_t}{R}$, we have

$$F'(k_{t+1}) + \gamma(\frac{i_{t+1}}{k_{t+1}})^2 - Rq_t + q_{t+1}(1-\varsigma) = 0$$

5. Combine the answer in (4) with the answer in (3) to write q_t as a function of q_{t+1} and k_{t+1} (get rid of i_{t+1}).

   Solution: Plugging (3) into (4), we have

$$q_t = \frac{1}{R}[F'(k_{t+1}) + \gamma(\frac{q_{t+1} - 1}{2\gamma})^2 + q_{t+1}(1-\varsigma)]$$

6. Combine the answer in (3) with the law of motion of capital to solve for k_{t+1} as a function of k_t and q_t.

   Solution: We use (3) to eliminate i_t in the law of motion of capital, and get

$$k_{t+1} = (1 - \varsigma + \frac{q_t - 1}{2\gamma})k_t$$

7. Now that you have two difference equations in q and k, what value of q_t sets $k_t = k_{t+1}$, as required in steady state? What values of k_t set $q_t = q_{t+1}$ as required in steady state?

   Solution: Using (6), we find that $k_t = k_{t+1}$ requires

$$\begin{aligned} 1 - \varsigma + \frac{q_t - 1}{2\gamma} &= 1 \\ q_t &= 2\gamma\varsigma + 1 \end{aligned}$$

   and using (5), we find that $q_t = q_{t+1}$ is satisfied when k_t solves

$$F'(k_t) = (R - 1 + \varsigma)q_t - \gamma\left(\frac{q_t - 1}{2\gamma}\right)^2$$

2 Optimization in Continuous Time

Let's solve the same problem in continuous time (note: $r = R - 1$, although this is not important to you solving the problem).

$$\begin{aligned} & \max_{c_t,k_t,a_t,i_t} \int_0^\infty e^{-rt} U(c_t) dt \\ & s.t. \\ \dot{a}_t &= -c_t + F(k_t) - i_t(1+\gamma \frac{i_t}{k_t}) + r a_t \\ \dot{k}_t &= -\varsigma k_t + i_t \end{aligned} \tag{2}$$

1. Write out the problem in Lagrange form. Let λ_t represent the multiplier for the first constraint and μ_t be the multiplier for the second constraint. Normalize each multiplier by e^{-rt}.

   Solution: We write

   $$L_t = \int_0^\infty e^{-rt}\{U(c_t) - \lambda_t[\dot{a}_t + c_t - F(k_t) + i_t(1+\gamma \frac{i_t}{k_t}) - r a_t] - \mu_t[\dot{k}_t + \varsigma k_t - i_t]\} dt$$

   Note that we have already normalized the multipliers; that is, we have used e.g. $\lambda_t e^{-rt} = \widetilde{\lambda}_t$ as the multiplier for the first constraint.

2. Before converting the equation into a present value Hamiltonian make the following change of variable: let $q_t = \mu_t / \lambda_t$, and rewrite the Lagrangian as a function of λ_t and q_t alone. Convert the equation into a present value Hamiltonian which omits $\dot{a}_t$ and $\dot{k}_t$ and includes $\dot{\lambda}_t$ and $\dot{q}_t$ instead.

   Solution: The change of variable gives us

   $$L_t = \int_0^\infty e^{-rt}\{U(c_t) - \lambda_t[\dot{a}_t + c_t - F(k_t) + i_t(1+\gamma \frac{i_t}{k_t}) - r a_t] - \lambda_t q_t[\dot{k}_t + \varsigma k_t - i_t]\} dt$$

   We then employ integration by parts on $\int_0^\infty e^{-rt} \lambda_t \dot{a}_t dt$ and $\int_0^\infty e^{-rt} \lambda_t q_t \dot{k}_t dt$, and drop the constant terms (we have ignored the assumptions on terminal conditions that allow us to do this; this is where they are important), to get our present-value Hamiltonian:

   $$H_t = e^{-rt}\{U(c_t) + a_t(\dot{\lambda}_t - r\lambda_t) - \lambda_t[c_t - F(k_t) + i_t(1+\gamma \frac{i_t}{k_t}) - r a_t] + k_t(\dot{\lambda}_t q_t + \lambda_t \dot{q}_t - r\lambda_t q_t) - \lambda_t q_t[\varsigma k_t - i_t]\}$$

3. Determine the first order conditions for c_t, k_t, a_t, and i_t.

Solution:

$$\begin{aligned}
c_t &: U'(c_t) - \lambda_t = 0 \\
k_t &: \lambda_t[F'(k_t) + \gamma(\frac{i_t}{k_t})^2] + \dot{\lambda}_t q_t + \lambda_t(\dot{q}_t - q_t(r+\varsigma)) = 0 \\
a_t &: \dot{\lambda}_t = 0 \\
i_t &: \lambda_t(1 + 2\gamma\frac{i_t}{k_t} - q_t) = 0
\end{aligned}$$

4. Combine the first order condition for a_t and the first order condition for k_t to cancel out all terms with λ_t in the equation for k_t.

   Solution: $\dot{\lambda}_t q_t$ drops out from the first order condition for a_t, so we are left with

   $$F'(k_t) + \gamma(\frac{i_t}{k_t})^2 + \dot{q}_t - q_t(r+\varsigma) = 0$$

5. Plug the first order condition for i_t into the equation just derived so that you are left with an expression of $\dot{q}_t$ as a function of q_t and k_t.

   Solution: The first order condition for i_t can be rearranged to yield

   $$\frac{i_t}{k_t} = \frac{q_t - 1}{2\gamma}$$

   and we can plug this into the answer from (4) to get

   $$\dot{q}_t = q_t(r+\varsigma) - F'(k_t) - \gamma(\frac{q_t - 1}{2\gamma})^2$$

6. Plug the first order condition for i_t into the law of motion for capital so that you have an expression for $\dot{k}_t$ as a function of q_t and k_t.

   Solution: We plug $\frac{i_t}{k_t} = \frac{q_t-1}{2\gamma}$ into the law of motion for capital to get

   $$\dot{k}_t = (\frac{q_t - 1}{2\gamma} - \varsigma)k_t$$

7. Now that you have two differential equations in q and k, what value of q_t sets $\dot{k}_t = 0$, as required in steady state? What values of k_t set $\dot{q}_t = 0$ as required in steady state?

   Solution: As in Question 1, $q_t = 2\gamma\varsigma + 1$ satisfies $\dot{k}_t = 0$. Similarly, for $\dot{q}_t = 0$, we have

   $$F'(k_t) = (r+\varsigma)q_t - \frac{1}{4\gamma}(q_t - 1)^2$$

   Notice (using $R - 1 = r$) that these are the same answers we had in Question 1.

3 Phase Diagrams

1. Using the result in part 7 of question 2, draw a phase diagram with q on the y axis and k on the x axis.

   Solution: The $\dot{k} = 0$ locus is horizontal at $q = 2\gamma\varsigma + 1$, and the $\dot{q} = 0$ locus is downward sloping near the steady state. See diagram at the end of these solutions for all parts of this question.

2. Is the system globally stable? Is it locally stable? If it is locally stable, where is the stable arm?

   Solution: The system is locally stable. For this problem set it was sufficient to see this graphically in the phase diagram, but note that if we assumed a particular functional form for $F(k)$ we could log-linearize and write the system as a first-order Taylor expansion, and find the eigenvector corresponding to the (one) negative eigenvalue of the 2x2 matrix in that expansion to actually identify the stable arm.

3. Now, assume that you are in the steady sate and there is an exogenous increase in γ. Describe the path of the economy, so describe in words what happens to k_t and to q_t. (HINT: when an exogenous shock occurs, we jump to the new stable arm immediately, and then follow it to the new steady state).

   Solution: See diagram, where γ has increased to $\underline{\gamma}$ and the new $\dot{k} = 0$ and $\dot{q} = 0$ loci are in bold. The jump to the new stable arm is the arrow pointing up from the initial steady state level of capital k_0^*, and then the economy moves along the stable arm to the new steady state at k_1^*. The idea is that in the immediate future, capital is fixed, but investment is not - so investment adjusts to as to move the economy onto the point (k_0^*, q) where q is determined by investment so as to meet the requirement that the economy be on the new stable arm.

4 Infinite Sums

1. Take the equation for q_t as a function of q_{t+1}, k_{t+1}, and i_t from problem 1 (make sure to keep the i_t term in there; otherwise you have a quadratic in q_{t+1}). Plug in recursively forward for q_{t+1} so you have q_t being equal to an infinite sum. What does this sum look like (you do not have to solve for anything explicitly, just plug in and explain intuitively what it looks like)? It turns out that q_t refers to Tobin's q here, so perhaps you can see from the result that this term tells us something about the future productive capacities of the economy.

Solution: We have

$$\begin{aligned}
q_t &= \frac{1}{R}\{F'(k_{t+1}) + \gamma(\frac{i_{t+1}}{k_{t+1}})^2 + q_{t+1}(1-\varsigma)\} \\
&= \frac{1}{R}\{F'(k_{t+1}) + \gamma(\frac{i_{t+1}}{k_{t+1}})^2 + [\frac{1}{R}\{F'(k_{t+2}) + \gamma(\frac{i_{t+2}}{k_{t+2}})^2 + q_{t+2}(1-\varsigma)\}](1-\varsigma)\} \\
&= \ldots \\
&= \sum_{s=0}^{\infty} \frac{(1-\varsigma)^s}{R^{s+1}}\{F'(k_{t+s+1}) + \gamma(\frac{i_{t+s+1}}{k_{t+s+1}})^2\}
\end{aligned}$$

(Note that the last term drops off so long as we assume that q_∞ does not explode). As for what this 'looks like' - the answer was basically given in the question; this infinite sum is a present discounted value of the future productive capacity of the capital stock

2. Take the equation of $\dot{q}_t$ as a function of q_t, k_t, and i_t from problem 2 (as before, do not plug in for i_t). Move all of the terms involving q to one side (Hint: you will be left with $\dot{q}_t - (r+\varsigma)q_t$ on one side). Multiply both sides by $e^{-(r+\varsigma)t}$ and integrate the two sides from t to ∞. You should have an expression (you do not have to solve explicitly for it) for q_t. What does this sum look like?

   Solution: The equation is

$$\dot{q}_t - q_t(r+\varsigma) = -F'(k_t) - \gamma(\frac{i_t}{k_t})^2$$

   Following the hints, we find that

$$\begin{aligned}
\int_t^\infty e^{-(r+\varsigma)\tau}[\dot{q}_\tau - q_\tau(r+\varsigma)]d\tau &= -\int_t^\infty e^{-(r+\varsigma)\tau}[F'(k_\tau) + \gamma(\frac{i_\tau}{k_\tau})^2]d\tau \\
q_t e^{-(r+\varsigma)\tau}\Big|_t^\infty &= -\int_t^\infty e^{-(r+\varsigma)\tau}[F'(k_\tau) + \gamma(\frac{i_\tau}{k_\tau})^2]d\tau \\
q_t &= e^{rt}\int_t^\infty e^{-(r+\varsigma)\tau}[F'(k_\tau) + \gamma(\frac{i_\tau}{k_\tau})^2]d\tau
\end{aligned}$$

   This is, of course, simply the continuous-time counterpart to the answer in part 1. Note that the point of this question was simply to illustrate how to use an integrating factor to change the differential equation in q into an expression for q that conveys the same information as the infinite sum in part 1.

3. Here is a different scenario. Imagine I have an exogenous income stream of $\overline{y}$ which I receive in every period so that $y_t = \overline{y}\forall t$. There are two interest rate in the economy. For periods $t \in [0, T_1]$ the interest rate is $1+r_1$. For periods $t \in (T_1, \infty)$, the interest rate is $1+r_2$.

(a) Assume we are in discrete time. Write out the present discounted value of my income stream from the standpoint of $t = 0$.

Solution: $\overline{y}\left[\sum_{t=0}^{T_1} \frac{1}{(1+r_1)^t} + \frac{1}{(1+r_1)^{T_1}} \sum_{s=1}^{\infty} \frac{1}{(1+r_2)^s}\right]$

(b) Assume we are in continuous time. Write out the present discounted value of my income stream from the standpoint of $t = 0$.

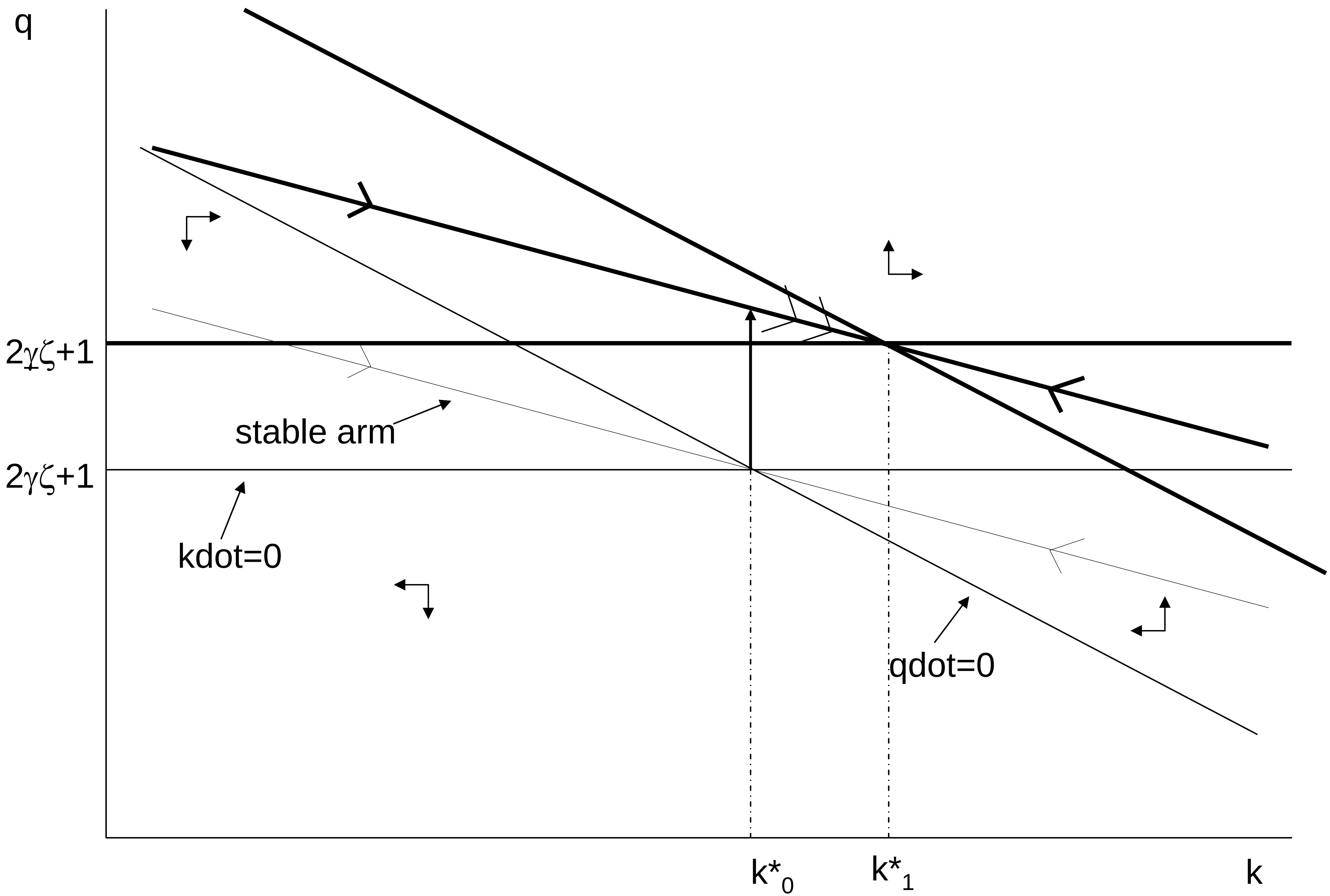
Solution: $\overline{y}\left[\int_0^{T_1} e^{-r_1 t} dt + \int_{T_1}^{\infty} e^{-r_1 T_1 - r_2 t} dt\right]$

(c) Assume the same setup in discrete time, but instead we have an arbitrary interest rate sequence $\{r_t\}_{t=0}^{\infty}$. What is the present discounted value of my income stream now?

Solution: $\overline{y}\left[\sum_{t=0}^{\infty} \frac{1}{\prod_{s=0}^{t}(1+r_s)}\right]$; note that we usually assume $r_0 = 0$.

(d) Assume the same setup in continuous time, but instead we have an arbitrary interest rate function $r(t)$. What is the present discounted value of my income stream now?

Solution: $\overline{y}\left[\int_0^{\infty} \exp\{-\int_0^t r(s)ds\}dt\right]$


q
2γζ+1
stable arm
2γζ+1
kdot=0
qdot=0
k*0
k*1
k