14.102 Problem Set 5

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1 Dynamic Optimization in a Deterministic Environment

Consider the below simple model of savings where a consumer maximizes the following program:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \tag{1}$$

$$\text{s.t. } \omega_t = e_t + R_t b_t,\ \omega_t = c_t + b_{t+1}, \tag{2}$$

$$\omega_0 > 0 \text{ given, } e_t = e\ \forall t,\ R_t = R\ \forall t, \tag{3}$$

$$\text{and } \sigma > 0,\ \sigma \neq 1 \tag{4}$$

e_t represents endowment, b_t represents bond holdings, and R_t is the interest rate. ω_t can be interpreted as cash in hand (money you make from endowment plus wealth).[1]

1. Note that $\omega_{t+1} = R_t(\omega_t - c_t) + e_{t+1}$ so that we can ignore $\{b_t\}_{t=0}^{\infty}$ altogether. Explain in words, why we can rewrite the problem in the following form Bellman Equation:

$$V(\omega_t) = \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(\omega_{t+1}) \right\} \tag{5}$$

2. Assuming that $V^{/}(\omega_t) > 0$ and $V^{//}(\omega_t) < 0$ derive the first order condition (FOC) and the envelope condition (EC). Combine the two to achieve the Euler Equation (EE) (a relationship between c_t and c_{t+1})

3. Assume that $e = 0$ so that there is no endowment stream. It turns out that in this case the value function will take the following form:

$$V(\omega_t) = a \frac{\omega_t^{1-\sigma}}{1-\sigma} \tag{6}$$

[1] Ignore the non-negativity constraint on consumption for the entire problem set.

Rewrite (5) substituting in (6) so that $V(\omega_{t+1})$ is a function of c_t and ω_t. Take the FOC of this new version of (5) with respect to c_t. This should give you a relationship between c_t and ω_t.

4. Plug in for c_t into the new version of (5). What you should have is an equation with ω_t on both sides. Show without solving explicitly for a, that the initial guess of for the value function is correct.

5. (Optional) Assume that $\beta R^{1-\sigma} < 1$ and solve for a. Determine c_t as a function of exogenous parameters.

6. Consider again the Euler Equation you derived in 1.2 ($e_t = e$ as before). What would be the path of consumption if $R\beta = 1$. Since β is the discount factor and R is the interest rate, what does this mean? (a simple increase/decrease with a story is what is required here). What about when $R\beta > 1$ or $R\beta < 1$? If you solved 1.5, do you get the same intuition if you examine the comparative static for c_t as a function of exogenous parameters as in 1.5?

2 Dynamic Optimization in a Stochastic Environment

Consider the same model of savings in a stochastic setting where a consumer maximizes the following program:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma} \tag{7}$$

$$\text{s.t. } \omega_t = e_t + R_t b_t,\ \omega_t = c_t + b_{t+1} \tag{8}$$

$$\omega_0 > 0 \text{ given, } e_t \sim \text{i.i.d}\left(e, \sigma_e^2\right),\ R_t = \text{i.i.d}\left(R, \sigma_R^2\right) \tag{9}$$

$$\text{and } \sigma > 0,\ \sigma \neq 1 \tag{10}$$

e_t represents endowment, b_t represents bond holdings, and R_t is the interest rate. ω_t can be interpreted as cash in hand (money you make from endowment plus wealth).

1. Note that $\omega_{t+1} = R_t(\omega_t - c_t) + e_{t+1}$ so that we can ignore $\{b_t\}_{t=0}^{\infty}$ altogether as before. Explain in words, why we can rewrite the problem in the following form Bellman Equation:

$$V(\omega_t) = \max_{c_t}\left\{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta EV(\omega_{t+1})\right\} \tag{11}$$

What would happen to the above value functions if e_t and R_t were each Markov as opposed to i.i.d? (This means that e_t's c.d.f. would depend on e_{t-1} and R_t's c.d.f. would depend on R_{t-1} where $e \perp R$. The answer here should be a very quick manipulation of (11)

2. Assuming that $V^{/}(\omega_t) > 0$ and $V^{//}(\omega_t) < 0$ derive the first order condition (FOC) and the envelope condition (EC). Combine the two to achieve the Euler Equation (EE) (a relationship between c_t and c_{t+1})

3. Assume that $e = 0$ and $\sigma_e^2 = 0$ so that there is no endowment stream. It turns out that in this case the value function will take the following form exactly as in the deterministic case:
$$V(\omega_t) = a\frac{\omega_t^{1-\sigma}}{1-\sigma} \tag{12}$$
Rewrite (11) substituting in (12) so that $V(\omega_{t+1})$ is a function of c_t and ω_t. Let $\widetilde{R}_{t+1} = \left(ER_{t+1}^{1-\sigma}\right)^{1/(1-\sigma)}$. Take the FOC of this new version of (11) with respect to c_t. This should give you a relationship between c_t and ω_t similar to the one you achieve in 1.3. It should be apparent that the solution to this value function is similar as in the deterministic case.

4. Consider again the Euler Equation you derived in 2.2 (and let e_t be stochastic as before). What would the path of consumption be now if $R\beta = 1$ and $\sigma_R^2 = 0$ so that the interest rate is non-stochastic? (Use Jensen's Inequality to relate c_t to $E(c_{t+1})$)