14.102 Problem Set 5 Solutions

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1 Dynamic Optimization in a Deterministic Environment

Consider the below simple model of savings where a consumer maximizes the following program:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \frac{c_t^{1-\sigma}}{1-\sigma} \tag{1}$$

$$\text{s.t. } \omega_t = e_t + R_t b_t,\ \omega_t = c_t + b_{t+1}, \tag{2}$$

$$\omega_0 > 0 \text{ given, } e_t = e\ \forall t,\ R_t = R\ \forall t, \tag{3}$$

$$\text{and } \sigma > 0,\ \sigma \neq 1 \tag{4}$$

e_t represents endowment, b_t represents bond holdings, and R_t is the interest rate. ω_t can be interpreted as cash in hand (money you make from endowment plus wealth).[1]

Note: When I wrote the solutions I failed to notice that R is constant, which is why 't+1' is appended to R throughout. Sorry about that.

1. Note that $\omega_{t+1} = R_{t+1}(\omega_t - c_t) + e_{t+1}$ so that we can ignore $\{b_t\}_{t=0}^{\infty}$ altogether. Explain in words, why we can rewrite the problem in the following form Bellman Equation:

$$V(\omega_t) = \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V(\omega_{t+1}) \right\} \tag{5}$$

   Solution: The value function represents a discounted sum of future utilities given optimal decision making. It is a function only of the state variable - what you are given to work with; that you will make the right decisions is assumed. Writing the problem in this recursive manner implies that you are thinking of the problem in the following way: you begin today with a given level of wealth. You consume some, and the rest becomes the level of wealth that your tomorrow-self will begin the day with (plus interest, etc.). You assume that your tomorrow-self, and day-after-tomorrow-self, and so on, will act optimally, and so the only decision you have

[1] Ignore the non-negativity constraint on consumption for the entire problem set.

to make is how much wealth to pass on, and how much to consume. Thus, an infinite-period problem is reduced to one with only two - today, and 'the rest of time'. And correspondingly, there is now just one control variable - consumption today, which determines wealth tomorrow.

2. Assuming that $V^{/}(\omega_t) > 0$ and $V^{//}(\omega_t) < 0$ derive the first order condition (FOC) and the envelope condition (EC). Combine the two to achieve the Euler Equation (EE) (a relationship between c_t and c_{t+1})

   Solution: Note first that we can write the problem as

$$V(\omega_t) = \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta V\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right) \right\} \tag{6}$$

   Then the first order condition is

$$c_t^{-\sigma} = \beta R_{t+1} V'\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right) \tag{7}$$

   And the envelope condition is

$$V'(\omega_t) = \beta R_{t+1} V'\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right) \tag{8}$$

   So $V'(\omega_t) = c_t^{-\sigma}$, and recalling that $V(R_{t+1}(\omega_t - c_t) + e_{t+1}) = V(\omega_{t+1})$, we can derive the Euler Equation:

$$1 = \beta R_{t+1} \left(\frac{c_{t+1}}{c_t} \right)^{-\sigma} \tag{9}$$

3. Assume that $e = 0$ so that there is no endowment stream. It turns out that in this case the value function will take the following form:

$$V(\omega_t) = a \frac{\omega_t^{1-\sigma}}{1-\sigma} \tag{10}$$

   Rewrite (5) substituting in (10) so that $V(\omega_{t+1})$ is a function of c_t and ω_t. Take the FOC of this new version of (5) with respect to c_t. This should give you a relationship between c_t and ω_t.

   Solution: If there is no endowment, $\omega_{t+1} = R_{t+1}(\omega_t - c_t)$. Then, using the hint, we have $V(\omega_{t+1}) = aR_{t+1}^{1-\sigma}\frac{(\omega_t - c_t)^{1-\sigma}}{1-\sigma}$, and we can rewrite the Bellman equation as

$$V(\omega_t) = \max_{c_t}\left\{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta a R_{t+1}^{1-\sigma}\frac{(\omega_t - c_t)^{1-\sigma}}{1-\sigma}\right\} \tag{11}$$

and the first order condition is

$$c_t^{-\sigma} = \beta a R_{t+1}^{1-\sigma}(\omega_t - c_t)^{-\sigma} \tag{12}$$

which, as promised, gives a relationship between c_t and ω_t.

4. Plug in for c_t into the new version of (5). What you should have is an equation with ω_t on both sides. Show without solving explicitly for a, that the initial guess of for the value function is correct.

   Solution: We have

$$c_t = \frac{\omega_t}{1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}} \tag{13}$$

   and so

$$\omega_t - c_t = \frac{(\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}\omega_t}{1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}} \tag{14}$$

   Plugging these in gives us the following Bellman equation:

$$\begin{aligned}
V(\omega_t) &= \max_{c_t}\left\{\left[\frac{1}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}} + \beta a R_{t+1}^{1-\sigma}\frac{[(\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}}\right]\frac{\omega_t^{1-\sigma}}{1-\sigma}\right\} \qquad (15) \\
&= \max_{c_t}\left\{\left[\frac{1}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}} + [(\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{\sigma}\frac{[(\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}}\right]\frac{\omega_t^{1-\sigma}}{1-\sigma}\right\} \\
&= \max_{c_t}\left\{\left[\frac{1}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}} + \frac{(\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}}{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{1-\sigma}}\right]\frac{\omega_t^{1-\sigma}}{1-\sigma}\right\} \\
&= \max_{c_t}\left\{[1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{\sigma}\frac{\omega_t^{1-\sigma}}{1-\sigma}\right\}
\end{aligned}$$

   and notice that we now have the Bellman equation in the form that we guessed it should be, so long as we can solve $a = [1 + (\beta a)^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{\sigma}$. There is a unique solution to this equation so long as $\beta R_{t+1}^{1-\sigma} < 1$.

5. (Optional) Assume that $\beta R_{t+1}^{1-\sigma} < 1$ and solve for a. Determine c_t as a function of exogenous parameters.

   Solution: Under this assumption, $a = [1 - \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}]^{-\sigma}$. We can plug this into our result from the last part to get

$$c_t = \omega_t(1 - \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}}) \tag{16}$$

   and

$$\omega_{t+1} = R_{t+1}(\omega_t - c_t) = (\beta R_{t+1})^{\frac{1}{\sigma}} \omega_t \tag{17}$$

   so

$$\omega_t = [(\beta R_{t+1})^{\frac{1}{\sigma}}]^t \omega_0 \tag{18}$$

   and so now, taking ω_0 as an exogenous 'parameter' (it is given), we have

$$c_t = (1 - \beta^{\frac{1}{\sigma}} R_{t+1}^{\frac{1-\sigma}{\sigma}})[(\beta R_{t+1})^{\frac{1}{\sigma}}]^t \omega_0 \tag{19}$$

6. Consider again the Euler Equation you derived in 1.2 ($e_t = e$ as before). What would be the path of consumption if $R\beta = 1$. Since β is the discount factor and R is the interest rate, what does this mean? (a simple increase/decrease with a story is what is required here). What about when $R\beta > 1$ or $R\beta < 1$? If you solved 1.5, do you get the same intuition if you examine the comparative static for c_t as a function of exogenous parameters as in 1.5?

   Solution: When $R\beta = 1$, $c_t = c_{t+1}$; the path of consumption is perfectly flat. When $R\beta > 1$, consumption is increasing; it is decreasing when $R\beta < 1$. The basic intuition is that R tells us what we get from saving, since whatever we don't consume grows at this rate. β, on the other hand, tells us how we discount the future. If, e.g., $R\beta > 1$, the former effect outweighs the latter, and we prefer to save more now and consume it later, hence an increasing growth path. The opposite holds when $R\beta < 1$, and when $R\beta = 1$ the two effects are perfectly balanced. Exactly the same algebra comes out of 1.5 if we assume that R is constant (which, as I noted above, I should have been doing the whole time!).

2 Dynamic Optimization in a Stochastic Environment

Consider the same model of savings in a stochastic setting where a consumer maximizes the following program:

$$\max_{\{c_t\}_{t=0}^{\infty}} E\sum_{t=0}^{\infty}\beta^t\frac{c_t^{1-\sigma}}{1-\sigma} \tag{20}$$

$$\text{s.t. } \omega_t = e_t + R_t b_t,\ \omega_t = c_t + b_{t+1} \tag{21}$$

$$\omega_0 > 0 \text{ given, } e_t \sim \text{i.i.d } \left(e, \sigma_e^2\right),\ R_t = \text{i.i.d } \left(R, \sigma_R^2\right) \tag{22}$$

$$\text{and } \sigma > 0,\ \sigma \neq 1 \tag{23}$$

e_t represents endowment, b_t represents bond holdings, and R_t is the interest rate. ω_t can be interpreted as cash in hand (money you make from endowment plus wealth).

1. Note that $\omega_{t+1} = R_t\left(\omega_t - c_t\right) + e_{t+1}$ so that we can ignore $\{b_t\}_{t=0}^{\infty}$ altogether as before. Explain in words, why we can rewrite the problem in the following form Bellman Equation:

$$V\left(\omega_t\right) = \max_{c_t}\left\{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta EV\left(\omega_{t+1}\right)\right\} \tag{24}$$

   What would happen to the above value functions if e_t and R_t were each Markov as opposed to i.i.d? (This means that e_t's c.d.f. would depend on e_{t-1} and R_t's c.d.f. would depend on R_{t-1} where $e \perp R$. The answer here should be a very quick manipulation of (24)

   Solution: The intuitive explanation for why we can write the problem in recursive form is exactly the same as it was in Question 1, with the added caveat that there is now uncertainty about $V\left(\omega_{t+1}\right)$, through the randomness characterizing the interest rate and endowment shock. The Markov adjustment would give

$$V\left(\omega_t, e_t, R_t\right) = \max_{c_t}\left\{\frac{c_t^{1-\sigma}}{1-\sigma} + \beta E[V\left(\omega_{t+1}, e_{t+1}, R_{t+1}\right)|e_t, R_t]\right\} \tag{25}$$

   which simply expresses the idea that now we are taking a *conditional* expectation, using information from the past period.

2. Assuming that $V^{/}\left(\omega_t\right) > 0$ and $V^{//}\left(\omega_t\right) < 0$ derive the first order condition (FOC) and the envelope condition (EC). Combine the two to achieve the Euler Equation (EE) (a relationship between c_t and c_{t+1})

Solution: Note first that we can write the problem as

$$V(\omega_t) = \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E V\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right) \right\} \tag{26}$$

Then the first order condition is

$$c_t^{-\sigma} = \beta E[R_{t+1} V'\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right)] \tag{27}$$

And the envelope condition is

$$V'(\omega_t) = \beta E[R_{t+1} V'\left(R_{t+1}(\omega_t - c_t) + e_{t+1}\right)] \tag{28}$$

So $V'(\omega_t) = c_t^{-\sigma}$, and recalling that $V(R_{t+1}(\omega_t - c_t) + e_{t+1}) = V(\omega_{t+1})$, we can derive the Euler Equation:

$$1 = \beta E\left[R_{t+1}\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}\right] \tag{29}$$

3. Assume that $e = 0$ and $\sigma_e^2 = 0$ so that there is no endowment stream. It turns out that in this case the value function will take the following form exactly as in the deterministic case:

$$V(\omega_t) = a\frac{\omega_t^{1-\sigma}}{1-\sigma} \tag{30}$$

Rewrite (24) substituting in (30) so that $V(\omega_{t+1})$ is a function of c_t and ω_t. Let $\widetilde{R}_{t+1} = \left(ER_{t+1}^{1-\sigma}\right)^{1/(1-\sigma)}$. Take the FOC of this new version of (24) with respect to c_t. This should give you a relationship between c_t and ω_t similar to the one you achieve in 1.3. It should be apparent that the solution to this value function is similar as in the deterministic case.

Solution: We have $V(\omega_{t+1}) = aR_{t+1}^{1-\sigma}\frac{(\omega_t - c_t)^{1-\sigma}}{1-\sigma}$, and we can rewrite the Bellman equation as

$$V(\omega_t) = \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta E[aR_{t+1}^{1-\sigma}\frac{(\omega_t - c_t)^{1-\sigma}}{1-\sigma}] \right\} \tag{31}$$

$$= \max_{c_t} \left\{ \frac{c_t^{1-\sigma}}{1-\sigma} + \beta \widetilde{R}_{t+1}^{1-\sigma} a \frac{(\omega_t - c_t)^{1-\sigma}}{1-\sigma} \right\} \tag{32}$$

and the first order condition is

$$c_t^{-\sigma} = \beta \widetilde{R}_{t+1}^{1-\sigma} a(\omega_t - c_t)^{-\sigma} \tag{33}$$

Note that the only uncertainty now is over the path of the interest rate.

4. Consider again the Euler Equation you derived in 2.2 (and let e_t be stochastic as before). What would the path of consumption be now if $R\beta = 1$ and $\sigma_R^2 = 0$ so that the interest rate is non-stochastic? (Use Jensen's Inequality to relate c_t to $E\left(c_{t+1}\right)$)

   Solution: The Euler Equation now becomes

$$c_t^{-\sigma} = E\left[c_{t+1}^{-\sigma}\right] > \left(E\left[c_{t+1}\right]\right)^{-\sigma} \tag{34}$$

   Where the last inequality follows from Jensen's inequality with $\sigma > 0$ (marginal utility is strictly convex). So we have $E_t[c_{t+1}] > c_t$, that is, we need $R\beta < 1$ to obtain zero expected growth in consumption.