14.102 Problem Set 1 Solutions

1. Lecture Notes Exercise 12: Show that $\mathbb{Q}$, the set of real rational numbers, does not have the least upper-bound property.

   Solution: To show this, we must show that there exists a set $E \subset \mathbb{Q}$ such that E is nonempty and bounded above, but for which $\sup E$ does not exist in $\mathbb{Q}$. One example of such a set E is $\{x \in \mathbb{Q} : x^2 < 2\}$. This is clearly nonempty and bounded above (by any member of $\mathbb{Q}$ greater than $\sqrt{2}$), and yet this set has no least upper bound, hence no supremum. For consider the set of all positive rationals x such that $x^2 > 2$. This is precisely the set of upper bounds of E in $\mathbb{Q}$ (note well, that, crucially, the real number x such that $x^2 = 2$ does not belong to $\mathbb{Q}$; indeed, we could be doing exactly this proof using any irrational number in place of $\sqrt{2}$). I claim that this set (call it F) has no least element: for every $p \in F$, we can find another $q \in F$ such that $q < p$.

   To do this, associate with every positive rational number p the number

   $$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}$$

   Note first that all such b are positive. Note also that subtracting $\sqrt{2}$ from both sides gives

   $$q - \sqrt{2} = \frac{(p - \sqrt{2})(2 - \sqrt{2})}{p + 2}$$

   So pick a $p \in F$. Then the associated q is positive, but because $p^2 - 2$ is also positive, we know that $q < p$. At the same time, knowing that $p - \sqrt{2}$ is positive implies that $q - \sqrt{2}$ is too, and so we have shown that $q \in F$, as we wanted to. Thus, we have shown that the set of upper bounds for E has no least element in $\mathbb{Q}$, which is the same as saying that $\sup E$ does not exist in $\mathbb{Q}$, implying that $\mathbb{Q}$ does not have the least-upper-bound property.

2. Is the set of real irrational numbers countable?

   Solution: No. This follows immediately from the facts (noted in class and shown in the lecture notes) that the reals are uncountable and the rationals are countable. Since the reals are the union (indeed, finite union) of the rationals and irrationals, the irrationals cannot be countable; if they were, we would have to conclude that the reals were countable (being the finite union of two countable sets), and they are not.

3. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$, define

   (a) $d_1(x, y) = (x - y)^2$

   (b) $d_2(x, y) = |x - 2y|$

   (c) $d_3(x, y) = \frac{|x-y|}{1+|x-y|}$

Determine for each of these whether it is a metric or not.

Solution:

(a) $d_1(x, y)$ is not a metric; it fails the triangle inequality. Consider $x = 2, y = 1, z = 0$; then $(x-y)^2 + (y-z)^2 = 2$, while $(x-z)^2 = 4$.

(b) $d_2(x, y)$ is also not a metric, for $d_2(1, 0) = 1 \neq 2 = d_2(0, 1)$.

(c) $d_3(x, y)$ is a metric. It is helpful to note that $|x - y|$ is itself a metric; it is therefore nonzero and positive for distinct x and y, zero for $x = y$, and $|x - y| = |y - x|$, which implies that $d_3(x, y)$ is itself nonzero and positive for distinct x and y, zero for $x = y$, and $d_3(x, y) = d_3(y, x)$. It remains to check the triangle inequality.

We want to show that

$$\frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|} \geq \frac{|x-z|}{1+|x-z|}$$

Let $|x - y| = a, |y - z| = b, |x - z| = c$. Then we want to show

$$\begin{aligned} \frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} &\geq 0 \\ \frac{a(1+b)(1+c) + b(1+a)(1+c) - c(1+a)(1+b)}{(1+a)(1+b)(1+c)} &\geq 0 \end{aligned}$$

Note that the denominator is positive because a, b, and c are all distances, and thus positive. So we just want to show that the numerator is positive. It reduces to $a + b - c + 2ab + abc$. The last two terms are clearly positive; moreover, because a, b, and c satisfy the triangle inequality we know that

$$a + b - c = |x - y| + |y - z| - |x - z| \geq 0$$

and so we know that the numerator is positive. This shows that $d_3(x, y)$ satisfies the triangle inequality; it is a metric.

4. Lecture Notes Exercise 37: Prove that the only limit point of a convergent sequence is its limit.

Solution: Consider a convergent sequence $x_n \to x$; we want to show that x is the only limit point of the sequence. So consider a limit point of the sequence, a. On the one hand, we know that for any $\varepsilon > 0$, and for any N, there is an $n \geq N$ such that $|x_n - a| < \varepsilon$, because a is a limit point. On the other hand, we also know that for any $\varepsilon > 0$, there exists an N - let's call it N_ε - such that for any $n \geq N_\varepsilon$, $|x_n - x| < \varepsilon$, because x is the limit of the sequence. Putting these two facts together with the triangle inequality, we know that there is an $n \geq N_\varepsilon$ such that $|x - a| \leq |x - x_n| + |x_n - a| = |x - x_n| + |a - x_n| < 2\varepsilon$, which establishes that $a = x$ for this limit point. Of course, the choice of a was arbitrary, and so this holds for all limit points, which is what we wanted to show.

5. Show that if a sequence $\{x_n\}$ satisfies the Cauchy criterion, then it is bounded.

   Solution: We know that $\{x_n\}$ is such that for every ε, there is an N such that $d(x_m, x_n) < \varepsilon$ if $m, n \geq N$. This is then true for $d(x_N, x_n)$ for $n \geq N$. This and the triangle inequality imply that for any $\varepsilon > 0$ and for all $n \geq N$, $d(x_n, 0) \leq d(x_N, 0) + d(x_N, x_n) < d(x_N, 0) + \varepsilon$, which is sufficient to bound the subsequence $\{x_n\}_{n=N}^{\infty}$ (take any number strictly greater than $|x_N|$). To show that the sequence as a whole is bounded, simply choose $C > \max\{\max\{|x_n|\}_{n=1}^{N-1}, |x_N|\}$, and it is the case that for all n, $C > |x_n|$.

6. Let E^o denote the set of all interior points of a set E (also called the interior of E).

   (a) Prove that E^o is always open.

   Solution: If E^o is open, then it is the case that for every point $x_0 \in E^o$, one can choose a small enough $\varepsilon > 0$ such that $B_\varepsilon(x_0) \subset E^o$ (not merely E, which is given by the fact that E^o consists entirely of interior points of E). Suppose this is not the case; suppose that there is some point $x_0 \in E^o$ such that $\forall \varepsilon > 0$, $B_\varepsilon(x_0)$ contains points which are in $E \backslash E^o$ (that is, points which are in E but not E^o - if there are no such points, this means merely that E had no interior points to begin with, so that E^o is the empty set, which is both open and closed, and we're done). Consider one of these points; call it x_1. Now, because x_1 is not an interior point of E, we know that $\forall \varepsilon > 0$, $B_\varepsilon(x_1)$ containts points which aren't in E. But now, pick any $\varepsilon > 0$, and we know that there is an $x_1 \in E \backslash E^o$ such that $d(x_0, x_1) < \frac{\varepsilon}{2}$, and a point $x_2 \notin E$ such that $d(x_1, x_2) < \frac{\varepsilon}{2}$, and then by the triangle inequality $d(x_0, x_2) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, which is to say that any open ball $B_\varepsilon(x_0)$ contains points which are not in E, a contradiction of the initial hypothesis that $x_0 \in E^o$. This completes the proof.

   (b) Prove that E is open if and only if $E = E^o$.

   Solution: We have just shown that E is open if $E = E^o$, because E^o is always open. It remains to show that $E = E^o$ if E is open. But this is immediate, since if E is open then it consists entirely of interior points.

   (c) If $G \subset E$ and G is open, prove that $G \subset E^o$.

   Solution: If G is open, then for any $g_0 \in G$ we can construct a small enough open ball $B_e(g_0) \subset G \subset E$, which implies that every g_0 is an interior point of E, i.e. that $G \subset E^o$.

   (d) Prove that the complement of E^o is the closure of the complement of E.

   Solution: Denote the closure of a set A as $\overline{A} \equiv A \cup A'$, where A' is the set of all limit points of A.

First we'll show that $(E^o)^c \subset \overline{E^c}$. Consider $x_0 \in (E^o)^c$. Then x_0 is not an interior point of E. If x_0 is not in E at all, then $x_0 \in E^c \subset \overline{E^c}$. If, on the other hand, $x_0 \in E$, then by virtue of the fact that it is not interior to E, we know that for any $\varepsilon > 0, \exists x_1 \in E^c$ such that $x_1 \in B_\varepsilon(x_0)$, which is precisely to say that x_0 is a limit point of E^c, $x_0 \in E^{c\prime} \subset \overline{E^c}$. Since this argument holds for all points of $(E^o)^c$, we have shown that $(E^o)^c \subset \overline{E^c}$, as desired.

Second we show the other set inclusion: $\overline{E^c} \subset (E^o)^c$. Once again we have two cases. If $x_0 \in E^c$, then it is clearly not an interior point of E, which is to say that $x_0 \in (E^o)^c$. On the other hand, if x_0 is a limit point of E^c, then for any $\varepsilon > 0, \exists x_1 \in E^c$ such that $x_1 \in B_\varepsilon(x_0)$, which also implies that x_0 is not an interior point of E. Once again this argument covers all points in $\overline{E^c}$, and so we have shown that $\overline{E^c} \subset (E^o)^c$. Together the two set inclusions imply set equality, which completes the proof that $(E^o)^c = \overline{E^c}$.

7. Let f be a continuous real function on a metric space X. Let $Z(f)$ be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

   Solution: We will show that $Z(f)$ contains all of its limit points. First note that if there are no $p \in X$ such that $f(p) = 0$, then $Z(f)$ is empty, and the empty set is closed. Suppose then that $Z(f)$ is nonempty, but has no limit points. Then all points of $Z(f)$ are isolated points, and we know of a set of isolated points that one can never center an open ball of any radius $\varepsilon > 0$ such that the open ball is contained in the set of isolated points - for this would imply that the point was not an isolated point at all, but a limit point. Thus, in this case $Z(f)$ is closed as well. Suppose therefore that $Z(f)$ is nonempty and does have limit points; again, we will show that all these limit points are contained in $Z(f)$.

   Suppose then that p_0 is a limit point of $Z(f)$. We want to show that $f(p_0) = 0$. Because f is continuous, we know that $\forall p, \in X$, and $\forall \varepsilon > 0$, $\exists \delta > 0$ such that if $p' \in B_\delta(p)$, then $f(p') \in B_\varepsilon(f(p))$. On the other hand, because p_0 is a limit point of $Z(f)$, we know that $\exists p \in Z(f)$ such that $p \neq p_0$ but $p \in B_\delta(p_0)$ (this would be true for any $\delta' > 0$; it is therefore true for the δ we used earlier). Putting the two together, we see that $f(p_0) \in B_\varepsilon(f(p))$ for arbitrarily small $\varepsilon > 0$. This argument holds for all limit points of $Z(f)$, and so is sufficient to complete the proof.

8. Prove that every Cobb-Douglas Function $F(x, y) = Ax^a y^b$ with A, a, and b all positive is quasiconcave.

   Solution: As noted in an email, I omitted the assumption that F is defined only over the positive quadrant $\{(x, y) : x > 0, y > 0\}$.

   It is easiest to solve this problem by noting that any concave function is quasiconcave, and that quasiconcavity is preserved under monotonic transformation. Therefore, if we can write a Cobb-Douglas function as a monotonic transformation of a concave function, then it must be quasiconcave. The first two facts were stated (though not proved) in class, and you could have taken them as given, but I'll go through all the steps here.

First, let's show that concavity implies quasiconcavity. Suppose that f defined on a convex subset U in $\mathbb{R}^n$ is concave. Such an f satisfies $f(\lambda x+(1-\lambda)y) \geq \lambda f(x)+(1-\lambda)f(y)$ for any $x, y \in U$ and $0 < \lambda < 1$. We will show that this means that $f(x) \geq f(y)$ implies $f(\lambda x+(1-\lambda)y) \geq f(y)$, again for any $x, y \in U$ and $0 < \lambda < 1$. So suppose that we have chosen x and y such that $f(x) \geq f(y)$. Then we have

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \geq \lambda f(y) + (1-\lambda)f(y) = f(y)$$

Second, we want to show that quasiconcavity is preserved under monotonic transformation. Suppose that g is a monotonically increasing transformation of f (i.e., $g = h \circ f$, where h is monotonically increasing - the argument where h, and therefore g, is decreasing is similar), and f is quasiconcave. Let us again suppose that we have chosen x and y such that $f(x) \geq f(y)$. Then

$$g(\lambda x + (1-\lambda)y) = (h \circ f)(\lambda x + (1-\lambda)y) \geq (h \circ f)(y) = g(y)$$

Finally, we want to show that we can always write a Cobb-Douglas function as a monotonic transformation of a concave function. The key here is to note that a Cobb-Douglas function with decreasing returns to scale is concave. This is most easily shown when we write

$$G(x, y) = \log A + a \log x + b \log y$$

Then the function is concave ($\frac{\partial^2 G}{\partial x^2} = -\frac{a}{x^2}, \frac{\partial^2 G}{\partial y^2} = -\frac{b}{y^2}$, and $\frac{\partial^2 G}{\partial x \partial y} = 0$; later we will give a more general version of the second order conditions for concavity, but in this case it is clear). This, by the way, is where the assumption that F is defined only over the positive quadrant comes into play; G is undefined for x or y negative. Note that we very often represent a Cobb-Douglas function in this way without even considering that in doing so, we may be using a concave function to represent one which is convex. But this is for good reason: we're not interested in preserving concavity, only quasiconcavity, for this is all we need to deliver the convex demand correspondences needed for utility theory.

Now, let us apply a monotonically increasing transformation to G - the exponential function: $\exp\{G(x, y)\} = Ax^a y^b = F(x, y)$. Thus, we can write any such Cobb-Douglas function as a monotonic transformation of a concave (also Cobb-Douglas) function, which proves that the function is quasiconcave. It is precisely because quasiconcavity is preserved under monotonic transformation that we can switch between the exponential and log-linear utility function without worrying about disturbing its ordinal properties.