14.102 Problem Set 2

Due Thursday, October 6, in class

1. Lecture Notes Exercise 105: Given an $m \times n$ matrix A, show that $S(B) \subseteq S(A)$ and $N(A') \subseteq N(B')$ whenever $B = AX$ for some matrix X. What is the geometric interpretation? *(Note: this is a repeat from last year's problem set; as such, the solution is right on the website. It is certainly worth doing, but the main reason I included it was to draw your attention to the result, which can be used to make part (e) of the next problem much less tedious.)*

2. Let $A = \begin{pmatrix} 1 & 3 & 0 \\ 2 & -1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \\ -1 & 1 \end{pmatrix}$

   (a) Find $C = AB$

   (b) Find $rank\ C$

   (c) Find det C

   (d) Find $D = BA$

   (e) Find $rank\ D$ (reminder: try to answer this using the result of problem 1 - without calculations)

   (f) Find det D

   (g) Is C invertible? If so, find C^{-1}

   (h) Is D invertible? If so, find D^{-1}

   (i) Find eigenvalues of C

   (j) Solve the following two linear systems (Hint: you will need no extra calculations!):

   i. $\begin{cases} \frac{1}{7}x + \frac{6}{7}y &= 1 \\ \frac{1}{7}x - \frac{1}{7}y &= 0 \end{cases}$

   ii. $\begin{cases} \frac{1}{7}u + \frac{6}{7}v &= 0 \\ \frac{1}{7}u - \frac{1}{7}v &= 1 \end{cases}$

3. Look at last year's problem set 1, #3, and its solution. It is good to understand the notion of changing bases, and of the coordinates of a vector with respect to a basis (we will use it again in discussing diagonalization). In particular, do Lecture Notes Exercise 114: what are the coordinates of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ with respect to the following bases?

   (a) $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

   (b) $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$

(d) $\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, assuming $\alpha\delta - \beta\gamma \neq 0$.

4. Prove Lemma 116 (note the hint in the lecture notes): Let $\{e_j\} = \{e_1, ..., e_n\}$ be a basis for $\mathbb{X}$, and let $\{b_j\} = \{b_1, ..., b_m\}$ be any set of vectors belonging to $\mathbb{X}$ with $m > n$. Then $\{b_j\}$ can not be linearly independent.

5. Lecture Notes Exercise 124: Using the 'fundamental theorem of algebra' and the fact that $rank(A) = rank(A')$, show that

$$\begin{aligned} rank(A) + null(A') &= m \\ null(A) - null(A') &= n - m \end{aligned}$$

6. Lecture Notes Exercise 129: Using the properties of transpose and inverse:

   (a) Prove that $A^{-k} = (A^k)^{-1}$

   (b) Consider the matrix $Z = X(X'X)^{-1}X'$ where X is an arbitrary $m \times n$ matrix. Under what conditions on X is Z well-defined? Show that Z is symmetric. Also show that $ZZ = Z$ (i.e., that Z is **idempotent**).

   (Note: this is another repeat, but this one I included simply because it really is worth doing - you will use these facts, and the techniques needed to prove them, a LOT in statistics and econometrics, so it would be helpful to get them down now.)

7. Lecture Notes Exercise 150: Show that, if $[A, b]$ is singular, then and only then $X^* \neq \emptyset$, and further $\dim(X^*) = null[A, b] - 1$.

8. Calculate e^A for A equal to

   (a) $\begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix}$ (hint: diagonalize!)

   (b) $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 6 \\ 0 & 0 & 0 \end{pmatrix}$ (hint: start with A^2, and recall that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$)

9. Lecture Notes Exercise 182: Let X be an $m \times n$ matrix with $m \geq n$ and $rk(X) = n$. Show that $X'X$ is positive definite.

10. Lecture Notes Exercise 183: Show that a positive definite matrix is non-singular.

    (Conclude from the past two exercises that so long as $m \geq n$ and $rk(X) = n$ - as you will generally assume when you estimate systems of equations - that you don't need to wonder whether the term $(X'X)^{-1}$ is defined.)