14.102 Problem Set 3

Due Tuesday, October 18, in class

1. Lecture Notes Exercise 208: Find $\int_a^b \log(t)dt$, where $0 < a < b$ are real numbers.

   Solution: This can be solved by integration by parts. Let $F(t) = t, F'(t) = f(t) = 1, G(t) = \log(t), G'(t) = g(t) = \frac{1}{t}$; then

$$\begin{aligned}
\int_a^b \log(t)dt &= \int_a^b f(t)G(t)dt = [F(b)G(b) - F(a)G(a)] - \int_a^b F(t)g(t)dt \\
&= [b\log(b) - a\log(a) - \int_a^b 1 \cdot dt \\
&= (b\log(b) - a\log a) - (b - a) \\
&= b(\log(b) - 1) - a(\log(a) - 1)
\end{aligned}$$

2. (Sundaram 4.4, page 110) Find and classify all critical points (local maximum, local minimum, neither) of the following function: $f(x, y) = e^{2x}(x + y^2 + 2y)$. For local optima that you find figure out whether they are also global optima.

   Solution: Critical points solve $\begin{cases} e^{2x}(2x + 2y^2 + 4y + 1) & = & 0 \\ e^{2x}(2y + 2) & = & 0 \end{cases}$.From the second equation, $y = -1$ and then from the first equation $x = \frac{1}{2}$. There is only one critical point $(\frac{1}{2}, -1)$. We have $H(\frac{1}{2}, -1) = \begin{pmatrix} 2e & 0 \\ 0 & 2e \end{pmatrix}$, which is positive definite, so it is a local minimum. Moreover, it is a global minimum too, since $f(x, y)$ is bounded from below (why?).

3. (Simon and Blume 18.7, page 423): Find the max and min of $f(x, y, z) = yz + xz$ subject to $y^2 + z^2 = 1$ and $xz = 3$

   Solution: The Lagrangian is $L(x, y, z, \lambda_1, \lambda_2) = yz + xz - \lambda_1(y^2 + z^2 - 1) - \lambda_2(xz - 3)$. The first order conditions give us:

$$\begin{aligned}
z - \lambda_2 z &= 0 & (1)\\
z - 2\lambda_1 y &= 0 & (2)\\
y + x - 2\lambda_1 z - \lambda_2 x &= 0 & (3)\\
y^2 + z^2 - 1 &= 0 & (4)\\
xz - 3 &= 0 & (5)
\end{aligned}$$

   where the last two simply repeat the constraints.

(1) tell us that either $z = 0$ or $\lambda_2 = 1$. But we know from (5) that z cannot be zero, so $\lambda_2 = 1$. Now, plug this into (3) and we find that (2) and (3) tell us that

$$\begin{aligned} z - 2\lambda_1 y &= 0 \\ y - 2\lambda_1 z &= 0 \end{aligned}$$

There are two sets of solutions to this pair of equations - one in which $\lambda_1 = \frac{1}{2}$ and $y = z$, and one in which $\lambda_1 = -\frac{1}{2}$ and $y = -z$. We now plug this into our constraints ((4) and (5)) to obtain our four critical points: $(3\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}); (-3\sqrt{2}, -\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}); (3\sqrt{2}, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}); (-3\sqrt{2}, \frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$. The first two points are maximizers (yielding a value of $3\frac{1}{2}$ for the objective function), while the second two are minimizers (yielding $2\frac{1}{2}$).

4. (Simon and Blume 18.11, page 434): Maximize $f(x, y) = 2y^2 - x$, subject to $x^2 + y^2 \leq 1, x \geq 0, y \geq 0$.

   Solution: The Lagrangian is $L(x, y, \lambda_1, \lambda_2, \lambda_3) = 2y^2 - x - \lambda_1(x^2 + y^2 - 1) + \lambda_2 x + \lambda_3 y$. Note that the terms in the Lagrangian associated with the last two constraints enter positively; those constraints can be rewritten as $-x \leq 0$ and $-y \leq 0$. We have the first order conditions:

$$\begin{aligned} \frac{\partial L}{\partial x} &= -1 - 2\lambda_1 x + \lambda_2 = 0 \\ \frac{\partial L}{\partial y} &= 4y - 2\lambda_1 y + \lambda_3 = 0 \\ \lambda_1(x^2 + y^2 - 1) &= 0, \lambda_2 x = 0, \lambda_3 y = 0 \\ \lambda_1 &\geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0 \\ x^2 + y^2 &\leq 1 \\ x &\geq 0, y \geq 0 \end{aligned}$$

   The third row contains the complementary-slackness conditions, while the last two rows give the first order conditions with respect to the Lagrange multipliers, and simply restate the intitial constraints.

   We will begin by showing that the first constraint must bind. Note that Weierstraß applies here; the constraint set is simply the positive quadrant of the unit circle, which is a compact set. Now, suppose that the first constraint does not bind, so that $x^2 + y^2 < 1$. Then $\lambda_1 = 0$ to satisfy the c-s condition. This implies, from the first FOC, that $\lambda_2 = 1$, which in turn implies (again from the c-s conditions) that $x = 0$. But if this is the case, the problem reduces to that of maximizing $2y^2$, subject to the constraints that y must be nonnegative and that $y^2 < 1$ (still assuming that the first constraint does not bind). But this clearly has no maximum, as the function is increasing in y, and an increasing function has no maximum on an open set. Since we know we have a maximum, it cannot be the case that the first constraint is slack when it is achieved. Therefore, it must be the case that $x^2 + y^2 = 1$.

But now we are merely looking for the point on positive quadrant of the circle which maximizes a function which is increasing in y and decreasing in x. It is clear that this occurs at the point $(0, 1)$.

5. Let $F(x, y) = 2x^2 + 2y^2 + 8$ and $G(x, y) = x^2 + 2y^2 - 6x - 7$. Note: this problem will be much easier and less tedious if you stop now and think about what these functions 'look like'.

   (a) State the implicit function theorem. Find all points on the curve $G(x, y) = 0$ around which either y is not expressible as a function of x or x is not expressible as a function of y. Compute $y'(x)$ along the curve when $x = 2$.

   Solution: This curve is the intersection of the xy plane with the paraboloid described by G, which is an ellipse. To see this, note that we can write $G(x, y) = (x - 3)^2 + 2y^2 - 16 = 0$, which clearly defines an ellipse with center $(3, 0)$, and major axis on the x-axis, with length 8.

   According to the implicit function theorem, the curve does not define $y(x)$ where $\frac{\partial G}{\partial y} = 4y = 0$, or at $y = 0$. Similarly, $x(y)$ is not defined where $\frac{\partial G}{\partial x} = 2x - 6 = 0$, or $x = 3$. Notice that these points are the 'top', 'bottom', and 'sides' of our ellipse, where the curve goes just vertical and horizontal.

   The implicit function theorem also tells us that, so long as $y(x)$ is defined, $y'(x) = \frac{\partial y}{\partial x_i}(x_1^*, ..., x_n^*) = -\frac{\frac{\partial G}{\partial x_i}(x_1^*, ..., x_n^*, y^*)}{\frac{\partial G}{\partial y}(x_1^*, ..., x_n^*, y^*)}$. Here, we have $y'(x) = -\frac{2x-6}{4y} = \frac{1}{2y}$ when $x = 2$. Plugging $x = 2$ into $G(x, y) = 0$, we find that $y = \pm\sqrt{\frac{15}{2}}$, so $y'(x) = \pm\frac{\sqrt{30}}{30}$.

   (b) Find all unconstrained optima of F and G on $\mathbb{R}^2$. Is the Weierstraß theorem applicable?

   Weierstraß does not apply, because $\mathbb{R}^2$ is unbounded. It is clear that neither of these functions have maxima. Nevertheless, we can find minima.

   Observe first that $G(x, y)$ can be split into $f(x) = x^2 - 6x$ and $g(y) = 2y^2 - 7$. Both of these functions describe convex parabolae (parabolas? paraboleese? whatever), so it should be clear that the function will have no global max, that it will have a global min, and that G as a whole describes a paraboloid in $\mathbb{R}^3$. Taking first order conditions, we find that

$$\begin{aligned} 2x - 6 &= 0 \qquad (6) \\ 4y &= 0 \qquad (7) \end{aligned}$$

so the only critical point is $(3, 0)$. Since both second order conditions are positive, we again see that the function is convex, and that this is therefore a global minimum. Exactly the same method can be applied to F, which yields $(0, 0)$ as the sole critical point and global minimum.

(c) Maximize and minimize $d(x, y) = \sqrt{x^2 + y^2}$ subject to $G(x, y) \leq 0$. Does Weierstraß apply?

Solution: Weierstraß does apply, because the ellipse defined by the constraint is a compact set and the distance function is continuous.

The simplest way to solve this problem is to treat it first as an equality constrained problem, then as an inequality constrained problem. The first is a matter of finding which points on the ellipse described by $G = 0$, and the second of finding which points on or inside the ellipse, are closest to and furthest from the origin.

The ellipse described by $G(x, y) = 0$ clearly comes closest to the origin at the point $(-1, 0)$ and is furthest from it at $(7, 0)$. But we can also get the same answer using the standard Lagrangian method, noting that the objective function is $d(x, y) =. \sqrt{x^2 + y^2}$ (the formula for distance from the origin).

$$
\begin{aligned}
L(x, y, \lambda) &= \sqrt{x^2 + y^2} - \lambda(x^2 + 2y^2 - 6x - 7) && (8)\\
\frac{\partial L}{\partial x} &= x(x^2 + y^2)^{-\frac{1}{2}} - \lambda(2x - 6) = 0 && (9)\\
\frac{\partial L}{\partial y} &= y(x^2 + y^2)^{-\frac{1}{2}} - 4\lambda y = 0 && (10)\\
\frac{\partial L}{\partial \lambda} &= x^2 + 2y^2 - 6x - 7 = 0 && (11)
\end{aligned}
$$

We note that $y = 0$ satsfies the second FOC. Plugging this into the constraint, we find that two critical points are $(-1, 0)$ and $(7, 0)$. Technically, we've treated x as a function of y, and should therefore be concerned about the points where this is not defined (i.e., at $x = 3$). This would give us two more critical points: $(3, 2\sqrt{2})$ and $(3, -2\sqrt{2})$. If we have drawn the ellipse that $G(x, y)$ represents we know that we need not check these points, however - and indeed, we find that

$$
\begin{aligned}
F(-1, 0) &= 1 && (12)\\
F(7, 0) &= 49 && (13)\\
F(3, 2\sqrt{2}) &= F(3, -2\sqrt{2}) = \sqrt{17} && (14)
\end{aligned}
$$

So that $(-1, 0)$ is our minimizer and $(7, 0)$ is our maximizer for the equality constrained problem. Now we have to check all points in the ellipse, in addition to its boundary. When the constraint doesn't bind, we have the additional critical point $(0, 0)$ - the origin itself. Thus, the maximum occurs at $(7, 0)$, and the minimum at $(0, 0)$.

(d) Maximize and minimize $F(x, y)$ subject to $G(x, y) = 0$. Is the Weierstraß theorem applicable?

Solution: Note that $F(x, y) = 2[d(x, y)]^2 + 8$, a monotonic transformation of $d(x, y)$. Thus, the extremal points must be the same. Weierstraß is once again applicable, and the max and min are at the points we found in part (c) for the equality constrained problem.

(e) Maximize and minimize $F(x, y)$ subject to $G(x, y) \leq 0$. Is the Weierstraß theorem applicable?

Weierstraß is applicable, and the max and min are the same points we found in part (c) for the inequality constrained problem.

(f) Maximize and minimize $F(x, y)$ subject to $G(x, y) \geq 0$. Is the Weierstraß theorem applicable?

Weierstraß is no longer applicable, since the set outside the ellipse is not bounded. There exists no maximum (no point furthest from the origin), but the minimum occurs at $(-1, 0)$, the point which we earlier found was the point on the ellipse closest to the origin.

6. (Simon and Blume 20.1, page 493): Which of the following functions are homogeneous? What are the degrees of homogeneity of the homogeneous ones?

(a) $3x^5y + 2x^2y^4 - 3x^3y^3$

Solution: The function is homogenous of degree six: $f(tx, ty) = 3t^6x^5y + 2t^6x^2y^4 - 3t^6x^3y^3 = t^6 f(x, y)$.

(b) $x^{1/2}y^{-1/2} + 3xy^{-1} + 7$

Solution: The function is homogenous of degree zero.

(c) $x^{3/4}y^{1/4} + 6x$

Solution: The function is homogenous of degree one.

(d) $\frac{(x^2-y^2)}{(x^2+y^2)} + 3$

Solution: The function is homogenous of degree zero.

7. (Simon and Blume 20.6, page 493): Prove that if f and g are functions on $\mathbb{R}^n$ that are homegeneous of different degrees, then $f + g$ is not homogeneous.

Solution: Let f be homogenous of degree k; then $f(tx) = t^k f(x)$; similarly, if g is homogenous of degree $l \neq k$, then $g(tx) = t^l g(x)$. Then $(f + g)(tx) = f(tx) + g(tx) = t^k f(x) + t^l g(x)$, and so $f + g$ is not homogenous.

8. Many utility functions we work with exhibit diminishing marginal returns (i.e., they are concave in each of their arguments, $\frac{\partial^2 u}{\partial x_i^2} < 0$). Is this an ordinal property? Why or why not?

Solution: No; it is not invariant under monotonic transformation. For example, the function $\log(x)$ is concave in x, but the monotonic transformation $x^2 = [\exp\{\log(x)\}]^2$ is convex.

9. Show that the following functions are homothetic:

   (a) $e^{x^2y}e^{xy^2}$

   Solution: $e^{x^2y}e^{xy^2} = \exp\{x^2y + xy^2\}$ which is a monotonic transformation of the homogeneous (of degree 3) function $x^2y + xy^2$.

   (b) $2\log x + 3\log y$

   Solution: $2\log x + 3\log y = \log(x^2y^3)$ which is a monotonic transformation of the homogeneous (of degree 5) function x^2y^3.

   (c) $x^3y^6 + 15x^2y^4 + 75xy^2 + 125$

   Solution: $x^3y^6 + 15x^2y^4 + 75xy^2 + 125 = (xy^2 + 5)^3$, which is a monotonic transformation of $xy^2 + 5$, which is a monotonic transformation of the homogenous (of degree 3) function xy^2.