Review Questions for 14.102 Midterm
10/14/05

Note: For true/false questions you should either prove the statement or provide a counterexample.

1 Real Analysis

1. Give an example of a relation R that is transitive, but not symmetric.

2. Suppose S is an ordered set, $E \subset S$, and E is bounded above. Define the supremum of E.

3. List the properties of a distance function. Is the following statement true: 'if $d(\cdot, \cdot)$ is a distance, then $d'(x, y) = (d(x, y))^2$ is a distance'?

4. State the Separating Hyperplane Theorem. Is it true that for any two disjoint *closed* convex sets C_1 and C_2 there exists a hyperplane $H(p, a)$ such that $p \cdot x < a$ for all $x \in C_1$ and $p \cdot y > a$ for any $y \in C_2$?

5. Show that the intersection of (even infinitely many) closed sets is closed. Give an example of an infinite family of closed sets whose union is not closed.

6. Prove carefully that the sum of two convergent sequences is convergent and its limit is the sum of the limits.

7. Define a limit point of a sequence. Is it true that if A is a limit point of a sequence $\{a_n\}$ and B is a limit point of a sequence $\{b_n\}$ then $A + B$ is a limit point of sequence $\{a_n + b_n\}$?

8. Let $A = [-1; 0)$ and $B = (0, 1]$. Examine whether each of the following statements is true or false:

   (a) $A \cup B$ is compact.

   (b) $A + B = \{x + y | x \in A, y \in B\}$ is compact.

   (c) $A \cap B$ is compact.

9. Define

   $$f(x) = \begin{cases} 1 \text{ if } 0 \leq x \leq 1 \\ 0 \ \ \text{otherwise} \end{cases}$$

Find an open set O such that $f^{-1}(O)$ is not open and a closed set C such that $f^{-1}(C)$ is not closed.

10. Find all the limit points of the following sequence: 1,1,2,1,2,3,1,2,3,4,...

11. Show that any finite union of finite sets (say, n of them) is itself finite.

2 Linear Algebra

In what follows, unless otherwise noted, let A be an $m \times n$ real matrix.

1. Define the nullspace of A.

2. State the rank-nullity theorem.

3. Consider an $m \times n$ matrix A.

   (a) Let $B = A'A$. Give an upper bound for $rank(A)$. Give an upper bound for $rank(B)$.

   (b) Provide a definition for the eigenvalues and eigenvectors of B.

   (c) Are the eigenvectors of B necessarily orthogonal to each other?

   (d) Are the eigenvectors of B necessarily orthonormal? If not, show how you would find a pair of orthonormal eigenvectors.

4. Define a symmetric matrix. Is it true that the product of two symmetric matrices is a symmetric matrix?

5. True or false: for a matrix to be diagonalizable it is both necessary and sufficient that it have n distinct eigenvalues.

6. Consider the $m \times n$ system of equations $Ax = b$. Under what conditions does there exist **no** solution to the system?

7. Give the eigenvalues for the diagonal square matrix

$$\begin{bmatrix} a_1 & 0 & \ldots & 0 \\ 0 & a_2 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & a_n \end{bmatrix}$$

   where $a_i = 2^i$, $i = 1, 2, ..., n$.

8. Show that $S(A)$ and $N(A')$ are orthogonal subspaces, in the sense that $z \in S(A), u \in N(A') \Rightarrow z'u = 0$. Show further that $S(A) + N(A') = \mathbb{R}^m$, in the sense that for every $y \in \mathbb{R}^m$ there are vectors $z \in S(A)$ and $u \in N(A')$ such that $y = z + u$.

3 Optimization in $\mathbb{R}^n$

1. State the Weierstraß theorem. Is it true that any function that is differentiable on a compact set is bounded on that set?

2. Give an example of a set $X \subset \mathbb{R}^n$, and a function $f : X \to \mathbb{R}$, such that the conditions of the Weierstraß theorem do **not** hold, but such that f nevertheless attains a maximum and a minimum on X.

3. State the implicit function theorem. Find all points on the curve $x^4 - 2x^2y^2 + y^4 = 0$ around which either y is not expressible as a function of x or x is not expressible as a function of y. Compute $y'(x)$ along the curve at point $(1, -2)$.

4. Consider the problem of maximizing $f(x)$ subject to $h(x) \leq 0$. The Lagrangian is

$$L(x_1, ..., x_n, \lambda_1, ..., \lambda_k) = f(x_1, ..., x_n) - \sum_{i=1}^{k} \lambda_i h_i(x_1, ..., x_n) \quad (1)$$

   Explain in words why it must be the case that $\lambda_i \geq 0$ for $i = 1, ..., k$ at a local optimum (x^*, λ^*).

5. Alden and Nicole recently moved into an apartment in need of a few repairs. For instance, the shower, water heater, and gas lines all needed some work before they could be used. Let us denote repairs to the shower by x, repairs to the water heater by y, and repairs to the gas lines by z, where $x, y, z \in \mathbb{R}$, and let us denote Alden and Nicole's utility from repairs to the three by

$$U(x, y, z) = xyz$$

   Alden and Nicole are endowed with one dollar (they're both grad students), which they can spend on repairs to the shower, water heater, and gas lines (assume they get utility only from the consumption of these three goods). Assume that $p_x = p_y = p_z = 1$.

   (a) Write down the maximization problem faced by Alden and Nicole, including their constraints (assume that we do not allow negative consumption of x, y, or z[1]).

[1] In real life, Alden could, say, try to work on the gas lines himself, making matters worse, but we'll just ignore that.

(b) Solve the maximization problem by the Kuhn-Tucker (i.e., Lagrangian) method, writing down all first order conditions, including complementary slackness conditions.

(c) Extra Credit: What (in words) is the interpretation of the functional form of Alden and Nicole's utility function?

6. (Sundaram 6.12, page 171) A firm produces a single output y using three inputs x_1, x_2, x_3 in nonnegative quantities through the relationship $y = x_1(x_2 + x_3)$.The unit price of y is $p_y > 0$ while that of the input x_i is $w_i > 0, i = 1, 2, 3$.

   (a) Describe the firm's profit-maximization problem and derive the equations that define the critical points of the Lagrangian L in this problem.

   (b) Show that the Lagrangian L has multiple critical points for any choice of $(p_y, w_1, w_2, w_3) \in \mathbb{R}^4_{++}$.

   (c) Show that none of these critical points identifies a solution of the profit-maximization problem. Can you explain why this is the case?

7. (Sundaram 8.25, page 201) An agent who consumes three commodities has a utility function given by $u(x_1, x_2, x_3) = \sqrt[3]{x_1} + \min\{x_2, x_3\}$. Given an income of I and prices p_1, p_2, p_3, write down the consumer's utility-maximization problem. Can the Weierstraß and/or Kuhn-Tucker theorems be used to obtain and characterize a solution? Why or why not?

8. (a) Consider the Euclidean distance from the origin to the point (x, y) in $\mathbb{R}^2$: $d(x, y) = \sqrt{x^2 + y^2}$. Suppose $d(x, y)$ reaches its global maximum on a compact set X at the point (x^*, y^*), and suppose that $z \to h(z)$ is a monotonically increasing transformation. Where does $F(x, y) \equiv (h \circ d)(x, y)$ attain its global maximum on X?

   (b) Consider the function $G(x, y) = x^2 + 2y^2 - 6x - 7$. Find the maximum and minimum of G on $\mathbb{R}^2$, if any. Does the Weierstraß theorem apply?

   (c) Consider now the curve described by $G(x, y) = 0$. Where does this curve **not** implicitly define y as a function of x? Where does the curve **not** implicitly define x as a function of y? Find the slope of the curve when $x = 2$.

   (d) Where is the curve $G(x, y) = 0$ closest to the origin? Does Weierstraß apply?

(e) Where is the curve $G(x, y) = 0$ farthest from the origin? Does Weierstraß apply?

(f) Maximize and minimize $d(x, y) = \sqrt{x^2 + y^2}$ subject to $G(x, y) \leq 0$. Does Weierstraß apply?

(g) Maximize and minimize $F(x, y) = 2x^2 + 2y^2 + 8$ subject to $G(x, y) \leq 0$.