

14.12 Game Theory – Midterm I

ANSWERS

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Prof. Muhamet Yildiz

Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 25 points. Good luck!

1. Find all the Nash equilibria in the following game:

1\2	L	M	R
T	1,0	0,1	5,0
B	0,2	2,1	1,0

Answer: By inspection, there is no pure-strategy equilibrium in this game. There is one mixed strategy equilibrium. Since R is strictly dominated, player 2 will assign 0 probability to R. Let p and q be the equilibrium probabilities for strategies T and L, respectively; the probabilities for B and R are $1 - p$ and $1 - q$, respectively. If 1 plays T, his expected payoff is $q \cdot 1 + (1 - q) \cdot 0 = q$. If he plays B, his expected payoff is $2(1 - q)$. Since he assigns positive probabilities to both T and B, he must be indifferent between T and B. Hence, $q = 2(1 - q)$, i.e., $q = 2/3$. Similarly, for player 2, the expected payoffs from playing L and M are $2(1 - p)$ and 1, respectively. Hence, $2(1 - p) = 1$, i.e., $p = 1/2$.

2. Find all the pure strategies that are consistent with the common knowledge of rationality in the following game. (State the rationality/knowledge assumptions corresponding to each operation.)

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2
B	1,0	0,1	0,2

Answer:

- (a) 1. For player 1, M strictly dominates B. Since **Player 1 is rational**, he will not play B, and we eliminate this strategy:

1\2	L	M	R
T	1,1	0,4	2,2
M	2,4	2,1	1,2

2. Since **Player 2 knows that Player 1 is rational**, he will know that 1 will not play B. Given this, the mixed strategy that assigns probability 1/2 to each of the strategies L and M strictly dominates R. Since **Player 2 is rational**, in that case, he will not play R. We eliminate this strategy:

1\2	L	M
T	1,1	0,4
M	2,4	2,1

3. Since **Player 1** knows that **Player 2** is rational and that **Player 2** knows that **Player 1** is rational, he will know that 2 will not play R. Given this, M strictly dominates T. Since **Player 1** is rational, he will not play T, either. We are left with

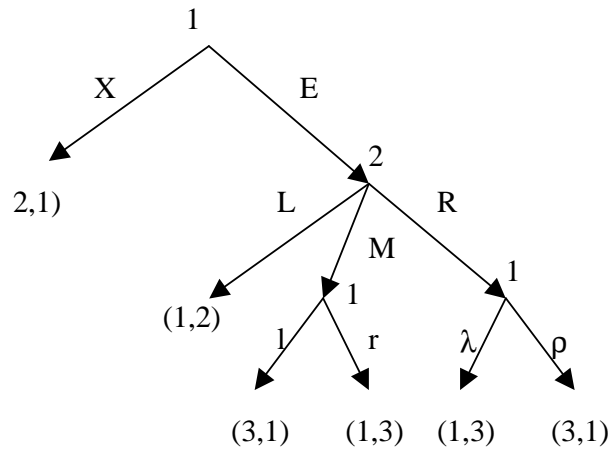
$$\begin{array}{c|cc}
 1 \backslash 2 & L & M \\
 \hline
 M & \boxed{2,4} & \boxed{2,1}
 \end{array}$$

4. Since **Player 2** knows that **Player 1** is rational, and that **Player 1** knows that **Player 2** knows that **Player 1** is rational, he will know that Player 1 will not play T or B. Given this, L strictly dominates M. Since **Player 2** is rational, he will not play M, either. He will play L.

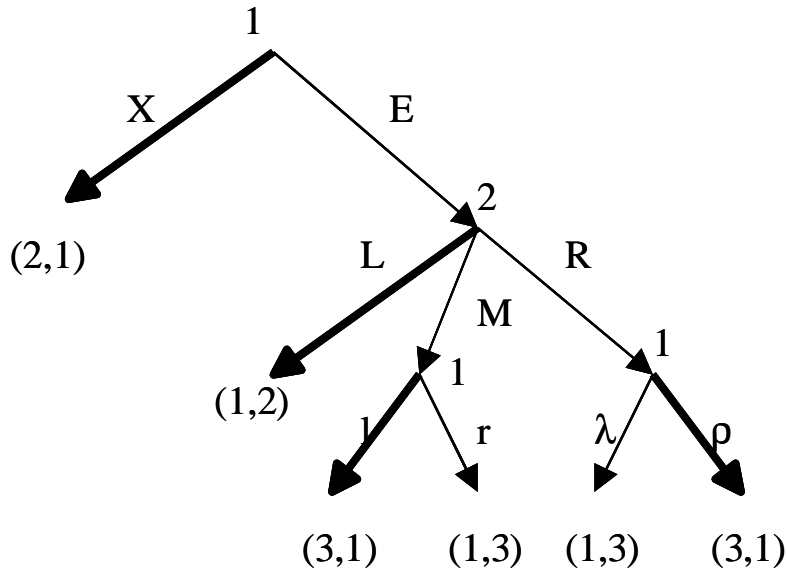
$$\begin{array}{c|c}
 1 \backslash 2 & L \\
 \hline
 M & \boxed{2,4}
 \end{array}$$

Thus, the only strategies that are consistent with the common knowledge of rationality are M for player 1 and L for player 2.

3. Consider the following extensive form game.



- (a) Using Backward Induction, compute an equilibrium of this game.



(b) Find the normal form representation of this game.

1\2	L	M	R
Xlλ	2,1	2,1	2,1
Xlρ	2,1	2,1	2,1
Xrλ	2,1	2,1	2,1
Xrρ	2,1	2,1	2,1
Elλ	1,2	3,1	1,3
Elρ	1,2	3,1	3,1
Erλ	1,2	1,3	1,3
Erρ	1,2	1,3	3,1

The points will be taken off from the people who did not distinguish the strategies that start with X from each other.

(c) Find all pure strategy Nash equilibria.

1\2	L	M	R
Xlλ	2,1	2,1	2,1
Xlρ	2,1	2,1	2,1
Xrλ	2,1	2,1	2,1
Xrρ	2,1	2,1	2,1
Elλ	1,2	3,1	1,3
Elρ	1,2	3,1	3,1
Erλ	1,2	1,3	1,3
Erρ	1,2	1,3	3,1

The Nash equilibria are (Xlλ,L), (Xlρ,L), (Xrλ,L), (Xrρ,L).

4. In this question you are asked to compute the rationalizable strategies in linear Bertrand-duopoly with discrete prices. We consider a world where the prices must be the positive multiples of cents, i.e.,

$$P = \{0.01, 0.02, \dots, 0.01n, \dots\}$$

is the set of all feasible prices. For each price $p \in P$, the demand is

$$Q(p) = \max\{1 - p, 0\}.$$

We have two firms $N = \{1, 2\}$, each with zero marginal cost. Simultaneously, each firm i sets a price $p_i \in P$. Observing the prices p_1 and p_2 , consumers buy from the firm with the lowest price; when the prices are equal, they divide their demand equally between the firms. Each firm i maximizes its own profit

$$\pi_i(p_1, p_2) = \begin{cases} p_i Q(p_i) & \text{if } p_i < p_j \\ p_i Q(p_i) / 2 & \text{if } p_i = p_j \\ 0 & \text{otherwise,} \end{cases}$$

where $j \neq i$.

- (a) Show that any price p greater than the monopoly price $p^{mon} = 0.5$ is strictly dominated by some strategy that assigns some probability $\epsilon > 0$ to the price $p^{min} = 0.01$ and probability $1 - \epsilon$ to the price $p^{mon} = 0.5$.

Answer: Take any player i and any price $p_i > p^{mon}$. We want to show that the mixed strategy σ^ϵ with $\sigma^\epsilon(p^{mon}) = 1 - \epsilon$ and $\sigma^\epsilon(p^{min}) = \epsilon$ strictly dominates p_i for some $\epsilon > 0$.

Take any strategy $p_j > p^{mon}$ of the other player j . We have

$$\pi_i(p_i, p_j) \leq p_i Q(p_i) = p_i(1 - p_i) \leq 0.51 \cdot 0.49 = 0.2499,$$

where the first inequality is by definition and the last inequality is due to the fact that $p_i \geq 0.51$. On the other hand,

$$\begin{aligned} \pi_i(\sigma^\epsilon, p_j) &= (1 - \epsilon)p^{mon}(1 - p^{mon}) + \epsilon p^{min}(1 - p^{min}) \\ &> (1 - \epsilon)p^{mon}(1 - p^{mon}) \\ &= 0.25(1 - \epsilon). \end{aligned}$$

Thus, $\pi_i(\sigma^\epsilon, p_j) > 0.2499 \geq \pi_i(p_i, p_j)$ whenever $0 < \epsilon \leq 0.0004$. Choose $\epsilon = 0.0004$.

Now, pick any $p_j \leq p^{mon}$. Since $p_i > p^{mon}$, we now have $\pi_i(p_i, p_j) = 0$. But

$$\pi_i(\sigma^\epsilon, p_j) = (1 - \epsilon)p^{mon}(1 - p^{mon}) + \epsilon p^{min}(1 - p^{min}) \geq \epsilon p^{min}(1 - p^{min}) > 0.$$

That is, $\pi_i(\sigma^\epsilon, p_j) > \pi_i(p_i, p_j)$. Therefore, σ^ϵ strictly dominates p_i .

- (b) Iteratively eliminating all the strictly dominated strategies, show that the only rationalizable strategy for a firm is $p^{\min} = 0.01$.

Answer: We have already eliminated the strategies that are larger than p^{\min} . At any iteration t assume that, for each player, the set of all remaining strategies are $P^t = \{0.01, 0.02, \dots, \bar{p}\}$ where $p^{\min} < \bar{p} \leq p^{\min}$. We want to show that \bar{p} is strictly dominated by the mixed strategy $\sigma_{\bar{p}}^{\epsilon}$ with $\sigma_{\bar{p}}^{\epsilon}(\bar{p} - 0.01) = 1 - \epsilon$ and $\sigma_{\bar{p}}^{\epsilon}(p^{\min}) = \epsilon$, and eliminate the strategy \bar{p} . This process will end when $P^s = \{0.01\}$, completing the proof. Now, for player i ,

$$\pi_i(\bar{p}, p_j) = \begin{cases} \bar{p}(1 - \bar{p})/2 & \text{if } p_j = \bar{p}, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$\begin{aligned} \pi_i(\sigma_{\bar{p}}^{\epsilon}, \bar{p}) &= (1 - \epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}(1 - p^{\min}) \\ &> (1 - \epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) \\ &= (1 - \epsilon)[\bar{p}(1 - \bar{p}) - 0.01(1 - 2\bar{p})]. \end{aligned}$$

Then, $\pi_i(\sigma_{\bar{p}}^{\epsilon}, \bar{p}) > \pi_i(\bar{p}, p_j)$ whenever

$$\epsilon \leq 1 - \frac{\bar{p}(1 - \bar{p})/2}{\bar{p}(1 - \bar{p}) - 0.01(1 - 2\bar{p})}.$$

But $\bar{p} \geq 0.02$, hence $0.01(1 - 2\bar{p}) < \bar{p}(1 - \bar{p})/2$, thus the right hand side is greater than 0. Choose

$$\epsilon = 1 - \frac{\bar{p}(1 - \bar{p})/2}{\bar{p}(1 - \bar{p}) - 0.01(1 - 2\bar{p})} > 0$$

so that $\pi_i(\sigma_{\bar{p}}^{\epsilon}, \bar{p}) > \pi_i(\bar{p}, p_j)$. Moreover, for any $p_j < \bar{p}$,

$$\begin{aligned} \pi_i(\sigma_{\bar{p}}^{\epsilon}, p_j) &= (1 - \epsilon)(\bar{p} - 0.01)(1 - \bar{p} + 0.01) + \epsilon p^{\min}(1 - p^{\min}) \\ &\geq \epsilon p^{\min}(1 - p^{\min}) > 0 = \pi_i(\bar{p}, p_j), \end{aligned}$$

showing that $\sigma_{\bar{p}}^{\epsilon}$ strictly dominates \bar{p} , and completing the proof.

- (c) What are the Nash equilibria of this game?

Answer: Since any Nash equilibrium is rationalizable, and since the only rationalizable strategy profile is (p^{\min}, p^{\min}) , the only Nash equilibrium is (p^{\min}, p^{\min}) . (Since this is a finite game, there is always a Nash equilibrium — possibly in mixed strategies.)