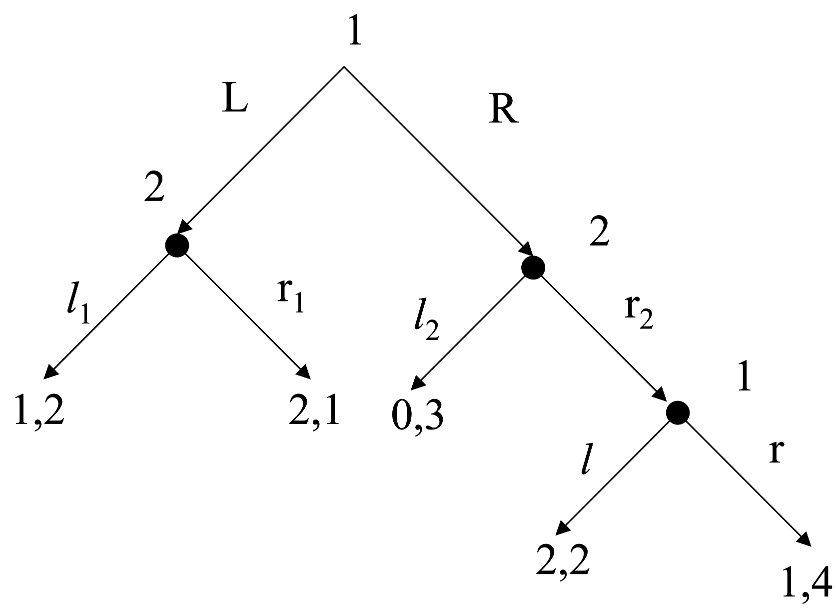
14.12 Game Theory – Midterm I
10/9/2002

Prof. Muhamet Yildiz

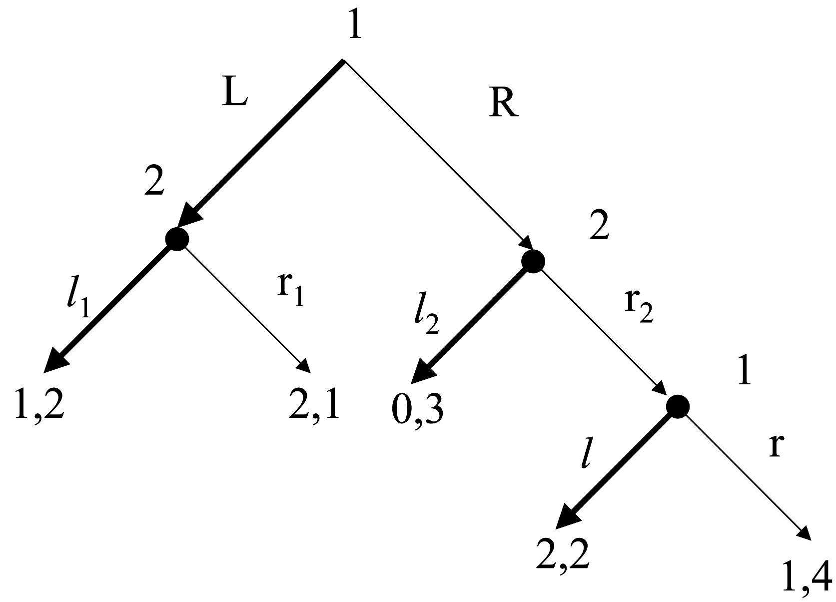
Instructions. This is an open book exam; you can use any written material. You have one hour and 20 minutes. Each question is 35 points. Good luck!

1. Consider the following game in extensive form.



(a) Apply backwards induction in this game. State the rationality/knowledge assumptions necessary for each step in this process.

The backwards induction outcome is as below. We first eliminate action r_1 for player 2, by assuming that *player 2 is sequentially rational* and hence will not play r_1, which is conditionally dominated by l_1.We also eliminate action r for player 1, assuming that *player 1 is sequentially rational*. This is because r is conditionally dominated by l. Second, assuming that *player 2 is sequentially rational* and that *player 2 knows that player 1 is sequentially rational*, we eliminate r_2. This is because, knowing that player 1 is sequentially rational, player 2 would know that 1 will not play r, and hence r_2 would lead to payoff of 2. Being sequentially rational she must play l_2. Finally, assuming that (i) *player 1 is sequentially rational*, (ii) *player 1 knows that player 2 is sequentially rational*, and (iii) *player 1 knows that player 2 knows that player 1 is sequentially rational*, we eliminate R. This is because (ii) and (iii) lead player 1 to conclude that 2 will play l_1 and l_2, and thus by (i) he plays L.



(b) Write this game in normal-form.

Each player has 4 strategies (named by the actions to be chosen).

	l_1l_2	l_1r_2	r_1l_2	r_1r_2
Ll	1,2	1,2	2,1	2,1
Lr	1,2	1,2	2,1	2,1
Rl	0,3	2,2	0,3	2,2
Rr	0,3	1,4	0,3	1,4

(c) Find all the rationalizable strategies in this game —use the normal form. State the rationality/knowledge assumptions necessary for each elimination.

First, Rr is strictly dominated by the mixed strategy that puts probability .5 on each of Ll and Rl. Assuming that *player 1 is rational*, we conclude that he would not play Rr. We eliminate Rr, so the game is reduced to

	l_1l_2	l_1r_2	r_1l_2	r_1r_2
Ll	1,2	1,2	2,1	2,1
Lr	1,2	1,2	2,1	2,1
Rl	0,3	2,2	0,3	2,2

.

Now r_1r_2 is strictly dominated by l_1l_2. Hence, assuming that (i) *player 2 is rational*, and that (ii) *player 2 knows that player 1 is rational*, we eliminate r_1r_2. This is because, by (ii), 2 knows that 1 will not play Rr, and hence by (i) she would not play r_1r_2. The game is reduced to

	l_1l_2	l_1r_2	r_1l_2
Ll	1,2	1,2	2,1
Lr	1,2	1,2	2,1
Rl	0,3	2,2	0,3

.

There is no strictly dominated strategy in the remaining game. Therefore, the all the remaining strategies are rationalizable.

(d) Comparing your answers to parts (a) and (c), briefly discuss whether or how the rationality assumptions for backwards induction and rationalizability differ.

Backwards induction gives us a much sharper prediction compared to that of rationalizability. This is because the notion of sequential rationality is much stronger than rationality itself.

(e) Find all the Nash equilibria in this game.

The only Nash equilibria are the strategy profiles in which player 1 mixes between the strategies Ll and Lr, and 2 mixes between $l_1 l_2$ and $l_1 r_2$, playing $l_1 l_2$ with higher probability:

$$NE = \{(\sigma_1, \sigma_2) \,|\sigma_1(Ll) + \sigma_1(Lr) = 1, \sigma_2(l_1 l_2) + \sigma_2(l_1 r_2) = 1, \sigma_2(l_1 r_2) \leq 1/2\}.$$

(If you found the pure strategy equilibria (namely, (Ll,$l_1 l_2$) and (Lr,$l_1 l_2$)), you will get most of the points.)

2. Consider two players A and B, who own a firm and want to dissolve their partnership. Each owns half of the firm. The value of the firm for players A and B are v_A and v_B, respectively, where $v_A > v_B > 0$. Player A sets a price p for half of the firm. Player B then decides whether to sell his share or to buy A's share at this price, p. If B decides to sell his share, then A owns the firm and pays p to B, yielding playoffs $v_A - p$ and p for players A and B, respectively. If B decides to buy, then B owns the firm and pays p to A, yielding playoffs p and $v_B - p$ for players A and B, respectively. All these are common knowledge. Find the subgame-perfect equilibrium of this game.

Given any price p, the best response of B will be

$$\begin{cases} \text{buy} & \text{if } v_B - p > p, \text{ i.e., if } p < v_B/2; \\ \text{sell} & \text{if } p > v_B/2; \\ \{\text{buy, sell}\} & \text{if } p = v_B/2. \end{cases}$$

In equilibrium, B must be selling at price $p = v_B/2$. This is because, if he were buying, then the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p \leq v_B/2; \\ v_A - p & \text{if } p > v_B/2, \end{cases}$$

which can be depicted as in Figure 1. Then, no price could maximize the payoff of A, inconsistent with equilibrium (where A maximizes his payoff given what he anticipates). Hence, the equilibrium strategy of B must be

$$\begin{cases} \text{buy} & \text{if } p < v_B/2; \\ \text{sell} & \text{if } p \geq v_B/2. \end{cases}$$

In that case, the payoff of A as a function of p would be

$$\begin{cases} p & \text{if } p < v_B/2; \\ v_A - p & \text{if } p \geq v_B/2, \end{cases}$$

which can be depicted as in Figure 2.

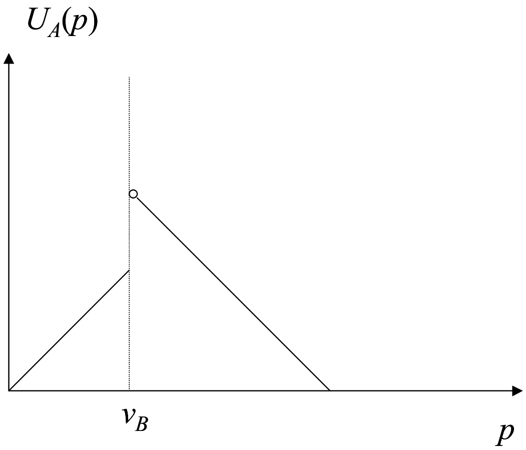
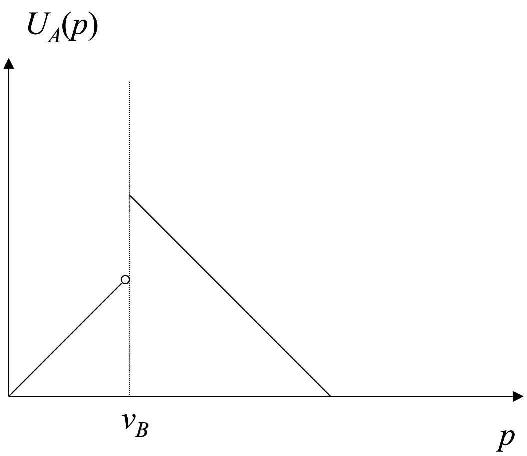


Figure 1:



This function is maximized at $p = v_B/2$. A sets the price as $p = v_B/2$.

3. Two start ups are competing for leadership in a software market. The leader wins, and the other loses. Each firm can invest some $x \in [0.001, 1]$ unit for research and development by paying cost of $x/4$. If a firm invests x units and the other firm invests y units, the former wins with probability $x/(x+y)$. Therefore, the payoff of the former start up will be

$$\frac{x}{x+y} - x/4.$$

All these are common knowledge.

   (a) Compute all pure strategy Nash equilibria.

   Call them as Firm 1 and Firm 2. Firm 1 maximizes

$$\frac{x}{x+y} - x/4$$

over x, and Firm 2 maximizes

$$\frac{y}{x+y} - y/4$$

over y. The best response function of Firm 1 as a function of y is given by

$$\begin{aligned} 0 &= \frac{\partial}{\partial x}\left(\frac{x}{x+y} - x/4\right) = \frac{\partial}{\partial x}\left(1 - \frac{y}{x+y} - x/4\right) \\ &= \frac{y}{(x+y)^2} - 1/4, \end{aligned}$$

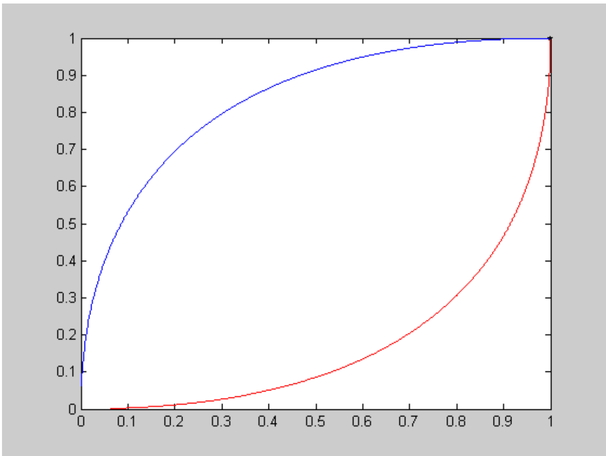
i.e.,

$$x^*(y) = 2\sqrt{y} - y.$$

Similarly, the best response function of Firm 2 is

$$y^*(x) = 2\sqrt{x} - x.$$

Note that $x^*(y) > y$ whenever $y < 1$. Therefore, the graphs of x^* and y^* intersect each other only at $x = y = 1$ —as shown in the figure below. Therefore, (1,1) is the only Nash equilibrium.



(b) Compute all rationalizable strategies.

(1,1) is the only rationalizable strategy profile. Since $y \geq y_0 \equiv 0.001$, then any strategy $x < x^*(y_0)$ is strictly dominated by $x_1 = x^*(y_0)$, and therefore eliminated. Write also $x_0 = y_0$ and $x_1 = y_1$. Now, the remaining strategy space of each player is $[x_1, 1]$. Note that $x_1 = x^*(.001) > 0.001 = x_0$. Now, similarly, we can eliminate any strategy $x < x_2 \equiv x^*(y_1)$. Applying this iteratively, after nth elimination we are left with a strategy space $[x_n, 1]$ where

$$x_n = 2\sqrt{x_{n-1}} - x_{n-1}$$

and $x_0 = .001$. It is clear from the figure that $x_n \to 1$ as $n \to \infty$. Hence in the limit we are left with strategy space $\{1\}$.

You do not need to do this: More formally,

$$x_n = 2\sqrt{x_{n-1}} - x_{n-1} > \sqrt{x_{n-1}} = x_{n-1}^{1/2}.$$

Hence,

$$1 > x_n > x_0^{(1/2)^{n-1}}.$$

Of course, as $n \to \infty$, $(1/2)^{n-1} \to 0$, and hence $x_0^{(1/2)^{n-1}} \to 1$. Therefore, $x_n \to 1$.