14.12 Economic Applications of Game Theory
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Problem 3 Solutions

1. (a) There are three subgames: [A] the subgame starting from Player 2's decision node after Player 1's choice of P2; [B] the subgame starting from Player 3's decision node following Player 1's P3; and [C] the entire game

For subgame [A], we can find three NE: two pure strategy NE - (x, l) (y, r) with the associated payoffs of $(3, 1)$ and $(1, 3)$ respectively, and one mixed strategy NE - $(\frac{3}{4}x + \frac{1}{4}y, \frac{1}{4}l + \frac{3}{4}r)$ with the payoff of $(\frac{3}{4}, \frac{3}{4})$. And the unique NE for subgame [B] is $(\frac{1}{2}a + \frac{1}{2}b, \frac{1}{2}L + \frac{1}{2}R)$ with the payoffs of $(\frac{3}{2}, \frac{3}{2})$.

Then for the entire game [C], we should consider the three following cases: i) for the first outcome of the subgame [A], Player 1 will choose P2; ii) for the second outcome, he will choose P3; iii) with the third outcome, he will choose P3.

Therefore, SPE for this game are

$$(P2x(\tfrac{1}{2}a + \tfrac{1}{2}b),\ l,\ \tfrac{1}{2}L + \tfrac{1}{2}R),$$
$$(P3y(\tfrac{1}{2}a + \tfrac{1}{2}b),\ r,\ \tfrac{1}{2}L + \tfrac{1}{2}R), \quad \text{and}$$
$$(P3(\tfrac{3}{4}x + \tfrac{1}{4}y)(\tfrac{1}{2}a + \tfrac{1}{2}b),\ \tfrac{1}{4}l + \tfrac{3}{4}r,\ \tfrac{1}{2}L + \tfrac{1}{2}R).$$

(b) Consider the node where the subgame [A] starts. At this node, if it is ever reached, Player 2 would believe Player 1's sequential rationality and foresee he will choose x, since the choice of y by Player 1 gives him at most 1 (which is less than 3/2, the minimum payoff he would expect to get if he has chosen P3 and given a best response to his belief about player 3's play), Player 2 will choose l. Therefore the second SPE is not consistent with Forward Induction. In a similar way, the third equilibrium also can be eliminated.

2. (a) The monopoly price is $p^m = 1/2$, and consider the following "trigger strategy".
- Start with playing p^m, and play p^m if everyone has played p^m all the previous periods.
- Play 0 if anyone has played something else different from p^m at least once.

By the single-deviation principle, it is sufficient to check if there are incentives to deviate only once, in order to check when this is actually a subgame perfect Nash Equilibrium.

First, check the incentive when no one has deviated from p^m before. If a firm plays p^m, as suggested by the strategy, then everyone will be playing p^m all the time in the future, and each firms's profit per period is given by

$$\frac{p^m(1-p^m)}{2} = \frac{1}{8}$$

and the present discounted sum of the profit sequence from today is

$$\frac{1}{8} + \delta \cdot \frac{1}{8} + \delta^2 \cdot \frac{1}{8} + \delta^3 \cdot \frac{1}{8} + \cdots = \frac{1}{1-\delta}\frac{1}{8}$$

If the firm is to deviate, the best it can do today is to set its price slightly less then p^m and get the the whole monopoly profit

$$p^m(1-p^m) = \frac{1}{4}$$

If the firm deviates, everyone will be playing 0 from the next period, yielding per period payoff of 0. Therefore, the present discounted sum of the profit sequence from today is

$$\frac{1}{4} + \delta \cdot 0 + \delta^2 \cdot 0 + \delta^3 \cdot 0 + \cdots = \frac{1}{4}$$

The firm has no incentive to deviate if and only if

$$\frac{1}{1-\delta}\frac{1}{8} \geq \frac{1}{4}$$

or

$$\delta \geq \frac{1}{2}$$

Second, let's check the incentive when someone has ever deviated from p^m before. In this case, a firm's action today doesn't affect its future payoffs because everyone will be playing 0 in the future no matter what happens today. Therefore, it is sufficient to check if the firm can increase the present profit by deviating . If it plays $p = 0$, as suggested the strategy, it gets a payoff of 0. If it deviates and plays $p > 0$, it gets again because all consumers buy from the other firm which is choosing $p = 0$. Therefore it can't benefit from deviating.

To conclude, the trigger strategy constitutes a subgame perfect Nash Equilibrium if $\delta \geq 1/2$.

(b) When there are n firms, we can again use the trigger strategy constructed in part (a). As we did in part (a) we check incentives to deviate before and after any deviation.

First, check the incentive to deviate when nobody has deviated from p^m before. If a firm plays p^m, as suggested by the strategy, then everyone will be playing p^m all the time in the future, and each firm's profit per period is

$$\frac{p^m(1-p^m)}{n} = \frac{1}{4n},$$

and the present discounted sum of the profit stream from today is

$$\frac{1}{4n} + \delta \cdot \frac{1}{4n} + \delta^2 \cdot \frac{1}{4n} + \cdots = \left(\frac{1}{1-\delta}\right)\frac{1}{4n}.$$

If the firm is to deviate, the best it can do today is to set a price slightly less then p^m, and get the whole monopoly profit

$$p^m(1-p^m) = \frac{1}{4}$$

If the firm does deviate, everyone will be playing 0 from the next period, yielding per period payoff of zero. Therefore, the present discounted sum of the profit streams from today is

$$\frac{1}{4} + \delta \cdot 0 + \delta^2 \cdot 0 + \cdots = \frac{1}{4}.$$

The firm has no incentive to deviate if and only if

$$\frac{1}{1-\delta}\frac{1}{4n} \geq \frac{1}{4}$$

or

$$\delta \geq \frac{n-1}{n}$$

To check the incentive when some one has never deviated from p^m before, the same argument as in part (a) holds.

To conclude, the trigger strategy constitutes a subgame perfect Nash Equilibrium if $\delta \geq \frac{n-1}{n}$.

3.

(a) To begin with, let's find the subgame perfect Nash Equilibrium of the stage game. We can do this by backward induction.

The long-run firm, observing the short-run firms's quantity x_t, chooses its quantity y_t to maximize its profit

$$\pi_t^L = y_t(1 - (x_t + y_t))$$

Solving the first order condition, the optimum is determined as

$$y_t^*(x_t^*) = \frac{1-x_t^*}{2}$$

The short-run firm chooses its quantity x_t to maximize its profit

$$\pi_t^S = x_t(1 - (x_t + y_t))$$

knowing that if it chooses x_t, the long-run firm reacts with

$$y_t^*(x_t) = \frac{1-x_t}{2}$$

Therefore, the short-run firm's objective function can be written as a function of x_t;

$$\pi_t^S = x_t(1 - (x_t + \frac{1-x_t}{2}))$$

Solving the first order condition, the optimum is

$$x_t^* = \frac{1}{2}$$

Therefore, the subgame perfect Nash Equilibrium of the stage game is

$$x_t^* = \frac{1}{2}, y_t^*(x_t) = \frac{1-x_t}{2} = \frac{1}{4}$$

Now let's solve for the subgame perfect Nash Equilibrium of the finitely repeated game.

Since it is a finite horizon game with perfect information, we can use backward induction. At the last period, $t = T$, the players don't care about the future, and they concern only about the payoffs of that period. Therefore they must play the subgame perfect Nash equilibrium of the stage game, regardless of what happened in the past. At time $t = T-1$, players know that their actions today don't affect tomorrow's outcome, so they will concern only about the payoff of that period. Therefore, they again must play the subgame perfect Nash equilibrium of the stage game. We can repeat this argument until we reach the first period. Therefore, the subgame perfect Nash equilibrium of the finitely repeated game is

$$x_t^* = \frac{1}{2}, y_t^*(x_t) = \frac{1-x_t}{2} = \frac{1}{4} \quad \text{for all } t$$

regardless of history.

(b) Consider the following trigger strategy.

[Long-run firm]
1) Start with playing the following strategy (*):
- $y(x) = 1/2$ if $x_t \leq 1/2$
- $y(x) = 1 - x_t$ if $x_t \geq 1/2$

Keep playing this strategy as long as it has not deviated from it.
2) Play $y_t(x_t) = \frac{1-x_t}{2}$ if it has deviated from (*) at least once.

[Short-run firms]
1) Play $x_t = \frac{1}{4}$ if the long-run firm has never deviated from (*) before.
2) Play $x_t = \frac{1}{2}$ if the long-run firm has deviated from (*) at least once.

To see this is actually a subgame perfect Nash equilibrium, let's check incentive to deviate.

First, consider the long-run firm's incentive when it has never deviated from (*) before.

Case 1: $x_t \leq 1/2$
If it follows the strategy and chooses $y_t = 1/2$, the present period profit of the long-run firm is

$$\frac{1}{2}(1 - (\frac{1}{2} + x_t)).$$

Starting from the next period, the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, and the long-run firm's per period profit is 1/8, and therefore the present discounted value of the profit stream is

$$\frac{1}{2}(1 - (\frac{1}{2} + x_t)) + \frac{\delta}{8(1-\delta)}.$$

If it is to deviate, the best it can do today is play

$$y_t = \frac{1-x_t}{2}$$

and get the payoff of

$$\frac{(1-x_t)^2}{4}.$$

However, starting from next period the outcome will be $x_t = 1/2$ and $y_t = 1/4$ and the long-run firm's per period profit is 1/6. Therefore the present discounted value of the profit stream is

$$\frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)}$$

If $\delta = 0.99$,

$$\frac{1}{2}(1-(\frac{1}{2}+x_t)) + \frac{\delta}{8(1-\delta)} > \frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)},$$

and therefore it is better to follow the equilibrium strategy than to deviate.

Case 2: $x_t \geq 1/2$

If it follows the strategy, then $p_t = 0$, and today's payoff is 0, and the outcome will be $x_t = 1/4$ and $y_t = 1/2$ every period, starting the next period. The long-run firm's per period profit is 1/8, and therefore the present discounted value of the profit sequence is

$$\frac{\delta}{8(1-\delta)}$$

If it is to deviate, the best it can do today is play $y_t = \frac{1-x_t}{2}$, and get the payoff of $\frac{(1-x_t)^2}{4}$. However, starting from next period, the outcome will be $x_t = 1/2$ and $y_t = 1/4$, and the long-run firm's per period profit is 1/16. Therefore the present discounted value of the profit stream is

$$\frac{(1-x_t)^2}{4} + \frac{\delta}{16(1-\delta)}.$$

This value is the largest when $x_t = 1/2$, and is equal to $\frac{1}{16} + \frac{\delta}{16(1-\delta)}$

If $\delta = 0.99$, it is better to follow the equilibrium strategy path than to deviate.

Second, consider the long-run firm's incentive when it has deviated from (*) before. According to the strategy profile, future outcomes don't depend on today's behavior. Therefore the long-run firm cares only about its payoff today. Actually, by following the strategy, it is taking best response to the short-run firm.

Finally,consider the short-run firm's incentives. Since they never care about future payoff, it must be playing a best response to the long-run firm's strategy, which is actually true.

(c) In the equilibrium we saw in part (b), the per period profits on the equilibrium path were 1/8 for the long-run firm and 1/16 for the sort-run firms.

If there is a subgame perfect Nash equilibrium where $x_t = y_t - 1/4$ on the equilibrium path, then the per period profits on the equilibrium path are 1/8 for the long-run firm and 1/8 for the short-run firm.

Construct the following trigger strategy, which is different from part (b) only in (*) where

$x_t = 1/4$:

[Long-run firm]
1) Start with playing the following strategy (*):
- $y_t(x_t) = 1/4$ if $x_t = 1/4$
- $y_t(x_t) = 1/2$ if $x_t \leq 1/2$ and $x_t \neq 1/4$
- $y_t(x_t) = 1/4$ if $x_t \geq 1/2$

Keep playing this strategy as long as it has not deviated from it.
2) Play $y_t^*(x_t) = (1 - x_t)/2$, if it has deviated from (*) at least once.

[Short-run firms]
1) Play $x_t = 1/4$ if the long-run firm has not deviated from (*) before.
2) Play $x_t = 1/2$ if the long-run firm has deviated from (*) at least once.

The incentive of the long-run firm when it is supposed to play (*) and $x_t = 1/4$ is satisfied because it gets current payoff of 1/8, which is the same as part (b), and what it can get by deviating is the same as part (b). Thus, the incentive problem of the long-run firm is the same as in part (b).

The incentive of the short-run firms is also satisfied because the payoff from following the strategy is larger than in part (b), and the payoff when deviating is the same as in part (b).

4. We can use the single deviation principle to check each case.

(a) No.
Since deviation is profitable given that other player is playing D ($6 > 5$), this cannot be a SPE.

(b) Yes.
Since deviation does not yield higher payoff than playing D ($0 < 1$) , this is a SPE.

(c) Yes.
Let's first check Player 1's incentive.
If he deviates and plays D when he is supposed to play C, then his payoff is less than that of the case he sticks to the equilibrium, that is,

$$6 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1 + \cdots = 6 + \tfrac{\delta}{1-\delta} < 5 + \delta \cdot 6 + \delta^2 \cdot 5 + \delta^3 \cdot 6 + \ldots = \tfrac{5}{1-\delta^2} + \tfrac{6\delta}{1-\delta^2}$$

And it is clear that there is no incentive for him to deviate when he is supposed to play D.

Next, check Player 2's incentive.
1) If she deviates to play D when Player 1 is supposed to play C, then her payoff is less than that of the case when she stays in the equilibrium, that is,

$$6 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1 + \cdots = 6 + \tfrac{\delta}{1-\delta} < 5 + \delta \cdot 0 + \delta^2 \cdot 5 + \delta^3 \cdot 0 + \ldots = \tfrac{5}{1-\delta^2}$$

2) If she deviates to play D when Player 1 is supposed to play D, then again her deviation is

not profitable since

$$1 + \delta \cdot 1 + \delta^2 \cdot 1 + \delta^3 \cdot 1 + \cdots = \frac{1}{1-\delta} < 0 + \delta \cdot 5 + \delta^2 \cdot 0 + \delta^3 \cdot 5 + \ldots = \frac{5\delta}{1-\delta^2}$$

Since deviation is not profitable for both players, this is SPE.

(d) No.

Suppose that a player has deviated in the previous period and sticks to the strategy, then from next period he will get $\delta \cdot 0 + \delta^2 \cdot 0 + \cdots + \delta^5 \cdot 0 + \cdots = 0$. While if he deviates from this (C, D) mode and plays D, he can get strictly positive payoff, making deviation profitable. Hence, this cannot be a SPE.

(e) No.

In DD mode, both players have no incentives to deviate.

In CC mode, suppose that a player has deviated to play D. If both players stick to the strategies described in the question, then he will get

$$6 + \{\delta \cdot 0 + \delta^2 \cdot 0 + \delta^3 \cdot 0 + \delta^4 \cdot 0 + \delta^5 \cdot 0\} + \{\delta^6 \cdot 6 + \delta^7 \cdot 6 + \cdots\} = 6 + \frac{6\delta^6}{1-\delta}$$

which is less than the payoff if he stays in the equilibrium.

Now consider P1 mode. During this punishment mode, if Player 1 deviates to play D although he is supposed to play C, he will get payoff greater than $\delta \cdot 0 + \delta^2 \cdot 0 + \delta^3 \cdot 0 + \delta^4 \cdot 0 + \delta^5 \cdot 0 = 0$, the payoff when he sticks to the strategy in the question. Hence, the deviation is profitable for Player 1 in P1 mode (Due to the symmetric nature of the game, we can apply the same argument to P2 mode.).

To conclude, this cannot be a SPE.