14.12 Economic Applications of Game Theory
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Problem Set 4 Solutions

1. (a)

- Action space: $A_1 = A_2 = \{B, S\}$

- Type Space: $T_1 = \{\alpha\}, T_2 = \{\beta_1, \beta_2\}$. Since Player 1 has no private information, we can model this so that her type can take only one value. Player 2 knows that the game above is played when his type is β_1 , and the game below is played when his type is β_2.

- Belief: Player *i*'s belief $\mu_i(t_j \mid t_i)$ is the probability that player *j*'s type is t_j conditional on that Player *i*'s type is t_i. In this model, since it is assumed that the types are independent,

$$\mu_1(\beta_1 \mid \alpha) = \mu_1(\beta_2 \mid \alpha) = 1/2,$$
$$\mu_1(\alpha \mid \beta_1) = \mu_1(\alpha \mid \beta_2) = 1$$

- vNM utility function: $U_i(a_1, a_2; t_1, t_2)$ is the vNM utility when Player 1's action is a_1, Player 2's action is a_2, Player 1's type is t_1 , and Player 2'a type is t_2.

$$U_1(B,B;\alpha,\beta_1) = 2;\ U_2(B,B;\alpha,\beta_1) = 1,$$
$$U_1(B,S;\alpha,\beta_1) = 0;\ U_2(B,S;\alpha,\beta_1) = 0,$$
$$U_1(S,B;\alpha,\beta_1) = 0;\ U_2(S,B;\alpha,\beta_1) = 0,$$
$$U_1(S,S;\alpha,\beta_1) = 1;\ U_2(S,S;\alpha,\beta_1) = 2,$$
$$U_1(B,B;\alpha,\beta_2) = 2;\ U_2(B,B;\alpha,\beta_2) = 0,$$
$$U_1(B,S;\alpha,\beta_2) = 0;\ U_2(B,S;\alpha,\beta_2) = 2,$$
$$U_1(S,B;\alpha,\beta_2) = 0;\ U_2(S,B;\alpha,\beta_2) = 1,$$
$$U_1(S,S;\alpha,\beta_2) = 1;\ U_2(S,S;\alpha,\beta_2) = 0.$$

(b) First consider Player 1's incentive. Since she doesn't know the game which is to be played, she wants to maximize her expected payoff.

If she plays B, with probability of ½ the top game is played and Player 2 chooses B and thus she gets a payoff of 2, and with probability of ½ the bottom game is played and Player 2 chooses S and thus she gets a payoff of 0. Therefore her expected payoff is 1. If she plays S, with probability of ½ the top game is played and Player 2 chooses B and thus she gets a payoff of 0, and with probability of ½ the bottom game is played and Player 2 chooses S and thus she gets a payoff of 0. Therefore her expected payoff is ½.

Therefore, B is actually Player 1's best response against Player 2's strategy.

Next consider Player 2's incentive. When he knows that the top game is being played, B is the best response given that Player 1 is choosing B. When he knows that the bottom game is being played, S is the best response given player 1 is choosing B. Therefore, choosing B when the top game is being played and choosing S when the bottom game is being played is actually Player 2's best response against Player 1's action.

Since both players are taking their best responses to each other, the strategy profile constitutes a Bayesian Nash Equilibrium.

2. Gibbons 3.2

Firms actions are the choice of quantities, and the amount of output can take any nonnegative values. Therefore, the strategy space is R_+ for each firm.

And since the information about demand is private to firm 1, we can model this fact as it has two types – high or low. One the other hand, firm 2 has only one type.

Let's find the Bayesian Nash equilibrium for this game.

First, consider the problem for firm. It knows the market demand, and wants to maximize its payoff for each state,

$$q_1(a) = \arg\max_{q_1} q_1(a - c - q_1 - q_2)$$

yielding,

$$q_1(a)\begin{cases} = q_1^H = (a_H - c - q_2)/2 & when\ a = a_H \\ = q_1^L = (a_L - c - q_2)/2 & when\ a = a_L \end{cases}$$

For firm 2, since it is uncertain about the market demand, it would wish to maximize its expected payoff,

$$q_2 = \arg\max_{q_2} \{\theta q_2(a_H - c - q_1^H - q_2) + (1-\theta)q_2(a_L - c - q_1^L - q_2)\}$$

or,

$$q_2 = \frac{\theta(a_H - q_1^H) + (1-\theta)(a_L - q_1^L) - c}{2}$$

Then, the equilibrium can be found by solving the above best responses simultaneously.

$$q_1^H = \frac{(3-\theta)a_H - (1-\theta)a_L - 2c}{6},$$

$$q_1^L = \frac{(2+\theta)a_L - \theta a_H - 2c}{6},$$

$$q_2 = \frac{\theta a_H + (1-\theta)a_L - c}{3}.$$

Since the output level is least in case for q_1^L, we need to assume $(2+\theta)a_L > \theta a_H + 2c$ in order for all equilibrium quantities to be positive.

3. Gibbons 3.3

Each player's action is the choice of price. A price can take any nonnegative real number. Therefore, the action space is R_+ for both players.

Player *i*'s type is her private information. In this model, b_i is player *i*'s type, and it is either b_H or. Therefore, the type space for each player is $\{b_H, b_L\}$.

Player *i*'s belief $\mu_i(b_j \mid b_i)$ is the probability that player *j*'s type is b_j conditional on that player *i*'s type is b_i. In this model, since it is assumed that the types are independent,

$$\mu_i(b_j \mid b_i) = \begin{cases} \theta & if \;\; b_j = b_H \\ 1-\theta & if \;\; b_j = b_L \end{cases}.$$

(vNM) utility in this model is the profit of each player (assuming that firms are risk neutral) as a function of the actions and types of both players:

$$U_i(p_i, p_j; b_i, b_j) = p_i(a - p_i - b_i p_j).$$

Player *i*'s strategies specify what actions to take for any realization of her type. In this model, it is a two dimensional vector $(p_i(b_H), p_i(b_L))$, where $p_i(b_H)$ is the price when its type is b_H and $p_i(b_L)$ is the price when its type is b_L. The strategy space is R_+^2 for each *i*.

A strategy profile $\{p_1^*(b_H), p_1^*(b_L), p_2^*(b_H), p_2^*(b_L)\}$ constitutes a Bayesian Nash equilibrium if each $p_i^*(b_i)$ is a best response, i.e., a maximizer of player *i*'s expected payoff, conditional on that her type is b_i and the opponent is choosing strategy $(p_j^*(b_H), p_j^*(b_L))$. That is,

$$\begin{aligned}
p_1^*(b_H) &= \arg\max_{p_1} \theta p_1(a - p_1 - b_H p_2^*(b_H)) + (1-\theta) p_1(a - p_1 - b_H p_2^*(b_H)) \\
p_1^*(b_L) &= \arg\max_{p_1} \theta p_1(a - p_1 - b_L p_2^*(b_H)) + (1-\theta) p_1(a - p_1 - b_L p_2^*(b_H)) \\
p_2^*(b_H) &= \arg\max_{p_2} \theta p_2(a - p_2 - b_H p_1^*(b_H)) + (1-\theta) p_2(a - p_2 - b_H p_1^*(b_H)) \\
p_2^*(b_L) &= \arg\max_{p_2} \theta p_2(a - p_2 - b_H p_1^*(b_H)) + (1-\theta) p_2(a - p_2 - b_H p_1^*(b_H))
\end{aligned}$$

Taking the first order conditions,

$$p_1^*(b_H) = \frac{a - b_H(\theta p_2^*(b_H) + (1-\theta) p_2^*(b_L))}{2},$$

$$p_1^*(b_L) = \frac{a - b_L(\theta p_2^*(b_H) + (1-\theta) p_2^*(b_L))}{2},$$

$$p_2^*(b_H) = \frac{a - b_H(\theta p_2^*(b_H) + (1-\theta) p_2^*(b_L))}{2},$$
$$p_2^*(b_L) = \frac{a - b_L(\theta p_2^*(b_H) + (1-\theta) p_2^*(b_L))}{2}.$$

Since the game is symmetric, let's look for a symmetric equilibrium where $p_1^*(b_H) = p_2^*(b_H) = p_H^*$, and $p_1^*(b_L) = p_2^*(b_L) = p_L^*$. Then the conditions are reduced to

$$p_H^* = \frac{a - b_H(\theta p_H^* + (1-\theta) p_L^*)}{2},$$
$$p_L^* = \frac{a - b_L(\theta p_H^* + (1-\theta) p_L^*)}{2}.$$

Solving these equations, we get

$$p_H^* = \frac{a}{2}(1 - \frac{b_H}{2 + \theta b_H + (1-\theta) b_L}),$$
$$p_L^* = \frac{a}{2}(1 - \frac{b_L}{2 + \theta b_H + (1-\theta) b_L}).$$

4. Gibbons 3.6

Let $i = 1,2,\ldots,n$ be the index of bidders, v_i bidder *i*'s valuation of the good, and b_i player *i*'s bid. We denote player *i*'s strategy by a function of $x_i(v_i)$, meaning that player i bids $b_i = x_i(v_i)$ when her valuation is v_i.

We want to show that the strategy profile

$$x_i(v_i) = \frac{(n-1)v_i}{n} \text{ for all } i$$

constitutes a Bayesian Nash equilibrium. Since the game is symmetric and strategy profile is all symmetric, it is sufficient to check one player's incentive because every player is facing the same incentive problem.

We will show that if player *i's v*aluation is v_i, and all other players are taking the strategy

$$x_j(v_j) = \frac{(n-1)v_j}{n}$$

then the bid which maximizes her expected payoff is

$$b_i = x_i(v_i) = \frac{(n-1)v_i}{n}$$

First, consider the probability of winning the auction is b_i. She wins if and only if all other players' bid are less than b_i, i.e.,

$$x_j(v_j) = \frac{(n-1)v_j}{n} \le b_i \text{ for all } j \ne i$$

This is equivalent to

$$v_j \le \frac{nb_i}{n-1} \quad \text{for all } j \ne i$$

Since

$$\Pr(v_j \le \frac{nb_i}{n-1}) = \frac{nb_i}{n-1}$$

for all *j* because v_j is uniformly distributed over [0,1],

$$\Pr(winning) = \Pr(v_j \le \frac{nb_i}{n-1}, \forall j \ne i) = \Pr(v_j \le \frac{nb_i}{n-1})^{n-1} = (\frac{nb_i}{n-1})^{n-1}.$$

Therefore, the expected payoff from the bidding b_i is

$$U_i(b_i) = (v_i - b_i)\Pr(winning) = (v_i - b_i)(\frac{nb_i}{n-1})^{n-1}$$

Taking the first order condition,

$$U'_i(b_i) = (v_i - b_i)(n-1)(\frac{nb_i}{n-1})^{n-2} - (\frac{nb_i}{n-1})^{n-1} = 0,$$

or

$$b_i = \frac{(n-1)v_i}{n}$$

Therefore, the strategy of player *i*,

$$b_i = x_i(v_i) = \frac{(n-1)v_i}{n}$$

is actually the best response to other players playing

$$x_j(v_j) = \frac{(n-1)v_j}{n}.$$

5. Gibbons 3.7

Bidder *i* would choose his bid $b = B(v_i)$ to maximize his expected payoff,

$$\begin{aligned}\pi_i &= (v_i - b_i)\Pr(b(v_j) < b_i) + \frac{1}{2}(v_i - b_i)\Pr(b(v_j) = b_i) \\ &= (v_i - b_i)\Pr(b(v_j) < b_i) \\ &= (v_i - b_i)F(B^{-1}(b_i)),\end{aligned}$$

where *F* represents the cumulative distribution function of valuations.

He would choose b_i such that $\partial\pi_i / \partial b_i = 0$. By differentiating π_i with respect to v_i, we obtain

$$\frac{d\pi_i}{dv_i} = \frac{\partial\pi_i}{\partial v_i} + \left(\frac{\partial\pi_i}{\partial b_i}\right)\frac{db_i}{dv_i} = \frac{\partial\pi_i}{\partial v_i}.$$ (Note this is actually the Envelope Theorem)

Thus, an optimally chosen bid b_i must satisfy

$$\frac{d\pi_i}{dv_i} = \frac{\partial\pi_i}{\partial v_i} = F(B^{-1}(b_i)).$$

Now, together with the symmetry assumption (if two bidders with the same valuation will submit the same bid), the equilibrium condition implies that bidder *i*'s optimal bid must be the bid implied by the decision rule *B* – in other words, at a equilibrium, $b_i = B(v_i)$. When we substitute this equilibrium condition into the above equation, we get

$$\frac{d\pi_i}{dv_i} = F(v_i)$$

We can solve the above differential equation for π_i by integrating (using the boundary condition, $B(0) = 0$),

$$\pi_i(v_i) = \int_0^{v_i} F(x)dx.$$

Then, combining this with the definition of expected payoff equation we can obtain each bidder's strategy

$$(v_i - b_i)F(v_i) = \int_0^{v_i} F(x)dx$$

or.

$$b_i = B(v_i) = v_i - \frac{\int_0^{v_i} F(x)dx}{F(v_i)} \quad \text{for } I = 1, 2.$$