14.12 Game Theory Lecture Notes
Lectures 15-18

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1 Cournot with Incomplete Information

- Demand:

$$P(Q) = a - Q$$

  where $Q = q_1 + q_2$.

- The marginal cost of Firm 1 = c_1; common knowledge.

- Firm 2's marginal cost:

$$\begin{array}{ll} c_H & \text{with probaility } \theta, \\ c_L & \text{with probaility } 1-\theta, \end{array}$$

  its private information.

- Each firm maximizes its expected profit.

How to find the Bayesian Nash Equilibrium?

Firm 2 has two possible types; and different actions will be chosen for the two different types.

$\{q_2(c_L), q_2(c_H)\}$

Suppose firm 2 is type high. Then, given the quantity q_1^* chosen by player, its problem is

$$\max_{q_2}(P - c_H)q_2 = [a - q_1 - q_2 - c_H]\, q_2.$$

Hence,

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \qquad (*)$$

Similarly, suppose firm 2 is low type:

$$\max_{q_2} \left[a - q_1^* - q_2 - c_H\right] q_2,$$

hence

$$q_2^*(c_L) = \frac{a - q_1^* - c_H}{2}. \tag{**}$$

Important Remark: The same level of q_1 in both cases. Why??

Firm 1's problem:

$$\max_{q_1} \theta \left[a - q_1 - q_2^*(c_H) - c\right] q_1 + (1-\theta) \left[a - q_1 - q_2^*(c_L) - c\right] q_1$$

$$q_1^* = \frac{\theta \left[a - q_2^*(c_H) - c\right] + (1-\theta) \left[a - q_2^*(c_L) - c\right]}{2} \tag{***}$$

Solve *, **, and *** for $q_1^*, q_2^*(c_L), q_2^*(c_H)$.

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{(1-\theta)(c_H - c_L)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} + \frac{\theta(c_H - c_L)}{6}$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1-\theta)c_L}{3}$$

Auctions Two bidders for a unique good.

v_i : valuation of bidder i.

Let us assume that v_i's are drawn independently from a uniform distribution over $[0,1]$. v_i is player i's private information. The game takes the form of both bidders submitting a bid, then the highest bidder wins and pays her bid.

Let b_i be player i's bid.

$v_i(b_1, b_2, v_1, v_2) = v_i - b_i$ if $b_i > b_j$

$\frac{v_i - b_i}{2}$ if $b_i = b_j$

0 if $b_i < b_j$

$$\max_{b_i} (v_i - b_i) Prob\{b_i > b_j(v_j) | \text{given beliefs of player i}) + \frac{1}{2}(v_i - b_i) Prob\{b_i = b_j(v_j)|...)$$

Let us first conjecture the form of the equilibrium: Conjecture: Symmetric and linear equilibrium

$$b = a + cv.$$

Then, $\frac{1}{2}(v_i - b_i)Prob\{b_i = b_j(v_j)|...) = 0$. Hence,

$$\max_{b_i}(v_i - b_i)Prob\{b_i \geq a + cv_j\} =$$
$$(v_i - b_i)Prob\{v_j \leq \frac{b_i - a}{c}\} = (v_i - b_i) \cdot \frac{(b_i - a)}{c}$$

FOC:

$$\begin{aligned} b_i &= \frac{v_i + a}{2} \qquad \text{if} \quad v_i \geq a \\ &= a \qquad \text{if} \quad v_i < a \end{aligned} \tag{1}$$

The best response b_i can be a linear strategy only if $a = 0$. Thus,

$$b_i = \frac{1}{2}v_i.$$

Double Auction Simultaneously, Seller names P_s and Buyer names P_b. If $P_b < P_s$, then no trade; if $P_b \geq P_s$ trade at price $p = \frac{P_b + P_s}{2}$.

Valuations are private information:

V_b uniform over $(0, 1)$

V_s uniform over $(0, 1)$ and independent from V_b

Strategies $P_b(V_b)$ and $P_s(V_s)$.

The buyer's problem is

$$\max_{P_b} E\left[V_b - \frac{P_b + P_s(V_s)}{2} : P_b \geq P_s(V_s)\right] =$$
$$\max_{P_b}\left[V_b - \frac{P_b + E[P_s(V_s)|P_b \geq P_s(V_s)]}{2}\right] \times Prob\{P_b \geq P_s(V_s)\}$$

where $E[P_s(V_s)|P_b \geq P_s(V_s)]$ is the expected seller bid *conditional* on P_b being greater than $P_s(V_s)$.

Similarly, the seller's problem is

$$\max_{P_s} E\left[\frac{P_s + P_b(V_b)}{2} - V_s : P_b(V_b) \geq P_s\right]] \quad =$$

$$\max_{P_s}\left[\frac{P_s + E[P_b(V_b)|P_b(V_b) \geq P_s}{2}\right] - V_s] \times Prob\{P_b(V_b) \quad \geq \quad P_s\}$$

Equilibrium is where $P_s(V_j)$ is a best response to $P_b(V_b)$ while $P_b(V_b)$ is a best response to $P_s(V_s)$.

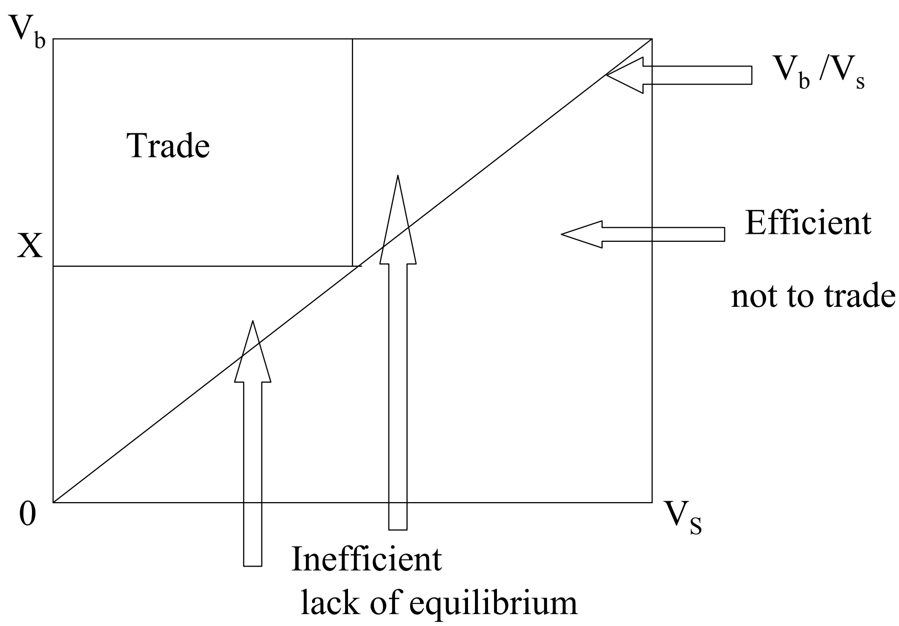
There are many Bayesian Nash Equilibria. Here is one.

$P_s = X \qquad \text{if} \qquad V_s \leq X$

$P_b = X \qquad \text{if} \qquad V_b \geq X.$

An equilibrium with "fixed" price.

Why is this an equilibrium? Because given $P_s = X$ if $V_s \leq X$, the buyer does not want to trade with $V_b < X$ and with $V_b > X, P_b = X$ is optimal.



Now construct an equilibrium with linear strategies:

$$p_b = a_b + c_b v_b$$

$$p_s = a_s + c_s v_s,$$

where a_b, a_s, c_b, and c_s are to be determined. Note that $p_b \geq p_s(v_s) = a_s + c_s v_s$ iff

$$v_s \leq \frac{p_b - a_s}{c_s}.$$

Likewise, $p_s \leq p_b(v_b) = a_b + c_b v_b$ iff

$$v_b \geq \frac{p_s - a_b}{c_b}.$$

Then, the buyer's problem is[1]

$$\begin{aligned}
&\max_{p_b} E\left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s)\right] \\
&= \max_{p_b} \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2}\right] dv_s \\
&= \max_{p_b} \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2}\right] dv_s \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{2}\int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2}\right) - \frac{c_s}{4}\left(\frac{p_b - a_s}{c_s}\right)^2 \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4}\right) \\
&= \max_{p_b} \frac{p_b - a_s}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right).
\end{aligned}$$

F.O.C.:

$$\frac{1}{c_s}\left(v_b - \frac{3p_b + a_s}{4}\right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

i.e.,

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \tag{2}$$

Similarly, the seller's problem is

[1] There is somewhat simpler way in to get the same outcome; see Gibbons.

$$\max_{p_s} E\left[\frac{p_s+p_b(v_b)}{2}-v_s : p_b(v_b)\geq p_s\right] = \max_{p_s}\int_{\frac{p_s-a_b}{c_b}}^{1}\left[\frac{p_s+a_b+c_bv_b}{2}-v_s\right]dv_b$$

$$= \max_{p_s}\left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2}-v_s\right]+\frac{c_b}{2}\int_{\frac{p_s-a_b}{c_b}}^{1} v_b dv_b$$

$$= \max_{p_s}\left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2}-v_s\right]+\frac{c_b}{4}\left(1-\left(\frac{p_s-a_b}{c_b}\right)^2\right)$$

$$= \max_{p_s}\left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{p_s+a_b}{2}-v_s+\frac{c_b}{4}+\frac{p_s-a_b}{4}\right]$$

$$= \max_{p_s}\left(1-\frac{p_s-a_b}{c_b}\right)\left[\frac{3p_s+a_b}{4}-v_s+\frac{c_b}{4}\right]$$

F.O.C.

$$-\frac{1}{c_b}\left[\frac{3p_s+a_b}{4}-v_s+\frac{1}{4}\right]+\frac{3}{4}\left(1-\frac{p_s-a_b}{c_b}\right)=0$$

i.e.,

$$-\left[\frac{3p_s+a_b}{4}-v_s+\frac{c_b}{4}\right]+\frac{3}{4}\left(c_b-(p_s-a_b)\right)=0,$$

i.e.,

$$\frac{3p_s}{2} = -\frac{a_b}{4}+v_s-\frac{c_b}{4}+\frac{3}{4}\left(c_b+a_b\right) = v_s+\frac{a_b+c_b}{2}$$

i.e.,

$$p_s = \frac{2}{3}v_s+\frac{a_b+c_b}{3}. \tag{3}$$

By (2), $a_b = a_s/3$, and by (3), $a_s = \frac{a_b}{3}+\frac{2}{9}$. Hence, $9a_s = a_s+2$, thus $a_s = 1/4$. Therefore, $a_b = 1/12$. The equilibrium is

$$p_b = \frac{2}{3}v_b+\frac{1}{12} \tag{4}$$

$$p_s = \frac{2}{3}v_s+\frac{1}{4}. \tag{5}$$