

Economic Applications with Incomplete Information

14.12 Game Theory
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Road Map

1. Cournot duopoly with Incomplete Information
2. A first price auction
3. A double auction
4. Quiz

1 Cournot duopoly with Incomplete Information

- Demand:

$$P(Q) = a - Q$$

where $Q = q_1 + q_2$.

- The marginal cost of Firm 1 = c ; common knowledge.
- Firm 2's marginal cost:
 c_H with probability θ ,
 c_L with probability $1 - \theta$;
its private information.
- Each firm maximizes its expected profit.

1.1 Bayesian Nash Equilibrium

Firm 2 of high type:

$$\max_{q_2} (P - c_H)q_2 = \max_{q_2} [a - q_1^* - q_2 - c_H] q_2.$$

$$q_2^*(c_H) = \frac{a - q_1^* - c_H}{2} \quad (*)$$

Firm 2 of low type:

$$\max_{q_2} [a - q_1^* - q_2 - c_L] q_2,$$

$$q_2^*(c_L) = \frac{a - q_1^* - c_L}{2}. \quad (**)$$

Firm 1:

$$\begin{aligned} \max_{q_1} \theta [a - q_1 - q_2^*(c_H) - c] q_1 \\ + (1 - \theta) [a - q_1 - q_2^*(c_L) - c] q_1 \end{aligned}$$

$$q_1^* = \frac{\theta [a - q_2^*(c_H) - c] + (1 - \theta) [a - q_2^*(c_L) - c]}{2} \quad (***)$$

Solve *, **, and *** for q_1^* , $q_2^*(c_L)$, $q_2^*(c_H)$.

$$\begin{pmatrix} q_1^* \\ q_2^*(c_H) \\ q_2^*(c_L) \end{pmatrix} = \begin{bmatrix} 2 & \theta & 1 - \theta \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}^{-1} \begin{pmatrix} a - c \\ a - c_H \\ a - c_L \end{pmatrix};$$

$$q_2^*(c_H) = \frac{a - 2c_H + c}{3} + \frac{(1 - \theta)(c_H - c_L)}{6}$$

$$q_2^*(c_L) = \frac{a - 2c_L + c}{3} - \frac{\theta(c_H - c_L)}{6}$$

$$q_1^* = \frac{a - 2c + \theta c_H + (1 - \theta)c_L}{3}$$

2 A First-price Auction

- One object, two bidders
- v_i = Valuation of bidder i ; iid with uniform distribution over $[0, 1]$.
- Simultaneously, each bidder i submits a bid b_i , then the highest bidder wins the object and pays her bid.
- The payoffs:

$$u_i(b_1, b_2, v_1, v_2) = \begin{cases} v_i - b_i & \text{if } b_i > b_j, \\ \frac{v_i - b_i}{2} & \text{if } b_i = b_j, \\ 0 & \text{if } b_i < b_j. \end{cases}$$

- Objective of i :

$$\max_{b_i} E [u_i(b_1, b_2, v_1, v_2)]$$

where

$$E [u_i] = (v_i - b_i) \Pr\{b_i > b_j(v_j)\} + \frac{1}{2}(v_i - b_i) \Pr\{b_i = b_j(v_j)\}$$

2.1 Symmetric, linear equilibrium

$$b_j = a + cv_j.$$

Then,

$$\Pr\{b_i = b_j(v_j)\} = 0.$$

Equilibrium condition: $a \leq b_i \leq v_i$.

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i) \Pr\{b_i \geq a + cv_j\} \\ &= (v_i - b_i) \Pr\{v_j \leq \frac{b_i - a}{c}\} \\ &= (v_i - b_i) \cdot \frac{b_i - a}{c}. \end{aligned}$$

FOC:

$$b_i = \begin{cases} \frac{v_i + a}{2} & \text{if } v_i \geq a \\ a & \text{if } v_i < a. \end{cases} \quad (1)$$

Therefore,

$$b_i = \frac{1}{2}v_i.$$

2.2 Any symmetric equilibrium

$$b_i = b(v_i)$$

Hence,

$$\begin{aligned} E[u_i] &= (v_i - b_i) \Pr\{b_i \geq b(v_j)\} \\ &= (v_i - b_i) \Pr\{v_j \leq b^{-1}(b_i)\} \\ &= (v_i - b_i) b^{-1}(b_i). \end{aligned}$$

FOC ($\partial/\partial b_i = 0$):

$$\begin{aligned} -b^{-1}(b_i) + (v_i - b_i) \frac{db^{-1}(b_i)}{db_i} &= 0 \\ -v_i + (v_i - b(v_i)) \frac{1}{b'(v_i)} &= 0 \\ b'(v_i) v_i + b(v_i) &= v_i \\ \frac{d[b(v_i) v_i]}{dv_i} &= v_i \end{aligned}$$

$$b(v_i) v_i = v_i^2/2 + \text{const.}$$

$$b(v_i) = v_i/2 + \text{const}/v_i.$$

$$b(0) = 0, \Rightarrow \text{const} = 0.$$

$$b(v_i) = v_i/2.$$

3 Double Auction

1. Simultaneously, Seller names p_s and Buyer names p_b .
 - a. If $p_b < p_s$, then no trade;
 - b. if $p_b \geq p_s$, trade at price $p = \frac{p_b + p_s}{2}$.
2. Valuations are private information: v_b, v_s iid w/ uniform on $[0, 1]$.
3. Payoffs:

$$u_b = \begin{cases} v_b - \frac{p_b + p_s}{2} & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$
$$u_s = \begin{cases} \frac{p_b + p_s}{2} - v_s & \text{if } p_b \geq p_s \\ 0 & \text{otherwise} \end{cases}$$

4. The buyer's problem:

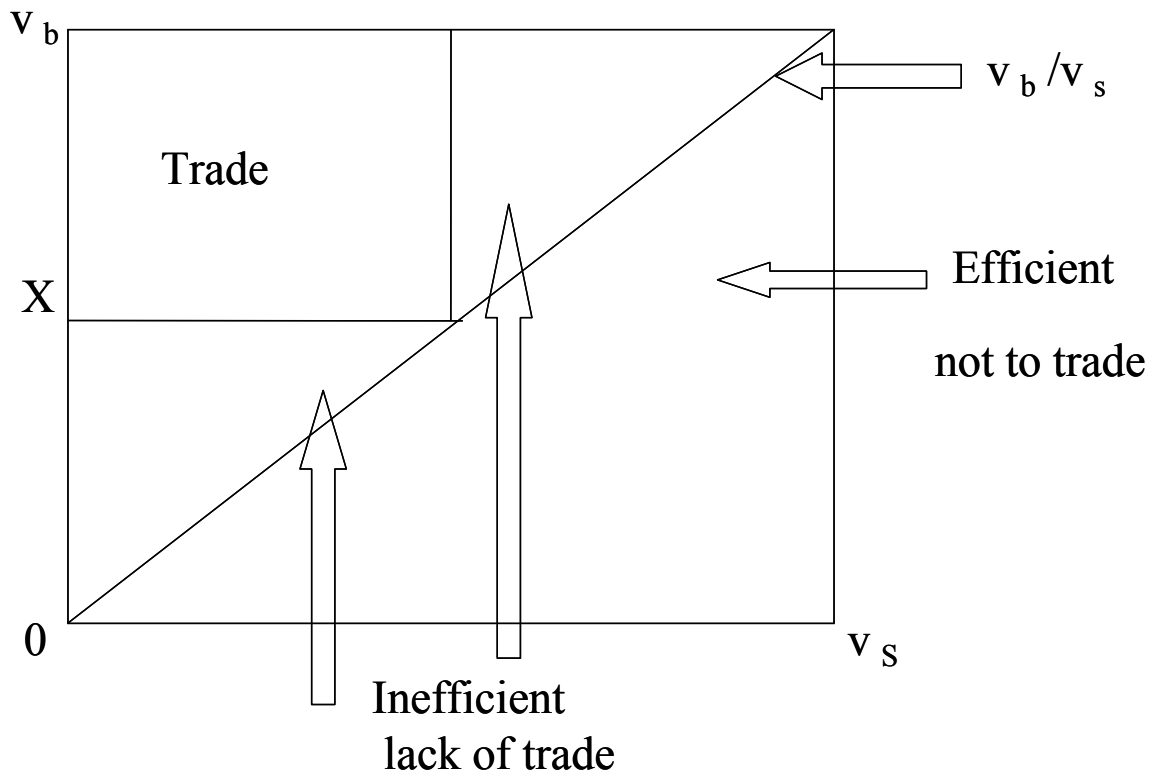
$$\max_{p_b} E \left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s) \right].$$

5. The seller's problem:

$$\max_{p_s} E \left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s \right].$$

An Equilibrium:

$$p_b = \begin{cases} X & \text{if } v_b \geq X \\ 0 & \text{otherwise} \end{cases},$$
$$p_s = \begin{cases} X & \text{if } v_s \leq X \\ 1 & \text{otherwise} \end{cases}.$$



Equilibrium with linear strategies:

$$p_b = a_b + c_b v_b$$

$$p_s = a_s + c_s v_s.$$

$$p_b \geq p_s(v_s) = a_s + c_s v_s \iff v_s \leq \frac{p_b - a_s}{c_s}.$$

$$p_s \leq p_b(v_b) = a_b + c_b v_b \iff v_b \geq \frac{p_s - a_b}{c_b}.$$

$$\begin{aligned}
E[u_b] &= E \left[v_b - \frac{p_b + p_s(v_s)}{2} : p_b \geq p_s(v_s) \right] \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + p_s(v_s)}{2} \right] dv_s \\
&= \int_0^{\frac{p_b - a_s}{c_s}} \left[v_b - \frac{p_b + a_s + c_s v_s}{2} \right] dv_s \\
&= \frac{p_b - a_s}{c_s} \left(v_b - \frac{p_b + a_s}{2} \right) - \frac{c_s}{2} \int_0^{\frac{p_b - a_s}{c_s}} v_s dv_s \\
&= \frac{p_b - a_s}{c_s} \left(v_b - \frac{p_b + a_s}{2} \right) - \frac{c_s}{4} \left(\frac{p_b - a_s}{c_s} \right)^2 \\
&= \frac{p_b - a_s}{c_s} \left(v_b - \frac{p_b + a_s}{2} - \frac{p_b - a_s}{4} \right) \\
&= \frac{p_b - a_s}{c_s} \left(v_b - \frac{3p_b + a_s}{4} \right).
\end{aligned}$$

F.O.C. ($\max_{p_b} E[u_b]$):

$$\frac{1}{c_s} \left(v_b - \frac{3p_b + a_s}{4} \right) - \frac{3(p_b - a_s)}{4c_s} = 0$$

i.e.,

$$p_b = \frac{2}{3}v_b + \frac{1}{3}a_s. \quad (2)$$

Similarly,

$$\begin{aligned}
E[u_s] &= E\left[\frac{p_s + p_b(v_b)}{2} - v_s : p_b(v_b) \geq p_s\right] \\
&= \int_{\frac{p_s - a_b}{c_b}}^1 \left[\frac{p_s + a_b + c_b v_b}{2} - v_s\right] dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s\right] + \frac{c_b}{2} \int_{\frac{p_s - a_b}{c_b}}^1 v_b dv_b \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s\right] \\
&\quad + \frac{c_b}{4} \left(1 - \left(\frac{p_s - a_b}{c_b}\right)^2\right) \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{p_s + a_b}{2} - v_s + \frac{c_b}{4} + \frac{p_s - a_b}{4}\right] \\
&= \left(1 - \frac{p_s - a_b}{c_b}\right) \left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4}\right]
\end{aligned}$$

F.O.C. ($\max_{p_s} E[u_s]$):

$$-\frac{1}{c_b} \left[\frac{3p_s + a_b}{4} - v_s + \frac{1}{4} \right] + \frac{3}{4} \left(1 - \frac{p_s - a_b}{c_b} \right) = 0$$

$$-\left[\frac{3p_s + a_b}{4} - v_s + \frac{c_b}{4} \right] + \frac{3}{4} (c_b - (p_s - a_b)) = 0,$$

$$\frac{3p_s}{2} = -\frac{a_b}{4} + v_s - \frac{c_b}{4} + \frac{3}{4} (c_b + a_b) = v_s + \frac{a_b + c_b}{2}$$

i.e.,

$$p_s = \frac{2}{3}v_s + \frac{a_b + c_b}{3}. \quad (3)$$

- $a_b = a_s/3$
- $a_s = a_b/3 + 2/9$
- Hence, $9a_s = a_s + 2$,
- $a_s = 1/4$; $a_b = 1/12$.

$$p_b = \frac{2}{3}v_b + \frac{1}{12} \quad (4)$$

$$p_s = \frac{2}{3}v_s + \frac{1}{4}. \quad (5)$$

We have trade iff

$$p_b \geq p_s$$

iff

$$\frac{2}{3}v_b + \frac{1}{12} \geq \frac{2}{3}v_s + \frac{1}{4}$$

iff

$$v_b - v_s \geq \frac{3}{2} \left(\frac{1}{4} - \frac{1}{12} \right) = \frac{31}{26} = \frac{1}{4}.$$

