

## 14.12 Solutions for Homework 1

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### Question 1

(a)

Remember that a binary relation  $\succeq$  is a *preference relation* if it is *complete* and *transitive*.

**Completeness** For all  $x, y \in X$ ,  $x \succeq y$  or  $y \succeq x$ .

**Transitivity** If  $x \succeq y$  and  $y \succeq z$ , then  $x \succeq z$ .

The relation defined here is complete, but it is not transitive. We can show this by a counterexample. Suppose  $x = 0$ ,  $y = .5$ , and  $z = 1$ . Then,  $x \succeq y$  because  $x = 0 = y - \frac{1}{2}$ , and  $y \succeq z$  because  $y = \frac{1}{2} = z - \frac{1}{2}$ , but we don't have  $x \succeq z$  because  $x = 0 < \frac{1}{2} = z - \frac{1}{2}$ . Therefore,  $\succeq$  is not a preference relation.

(b)

$$\begin{aligned} & x \succ y \\ \Leftrightarrow & x \geq y - 1/2 \text{ and } y < x - 1/2 \\ \Leftrightarrow & x > y + 1/2 \end{aligned}$$

Suppose  $x \succ y$  and  $y \succ z$ . Then,  $x > y + 1/2$  and  $y > z + 1/2$ . This implies that

$$x > y + 1/2 > (z + 1/2) + 1/2 > z + 1/2.$$

Therefore  $\succ$  is transitive.

We can show that  $\sim$  is not transitive by using the same counterexample used in (a). Suppose  $x = 0$ ,  $y = .5$ , and  $z = 1$ . Since  $x = 0 = y - \frac{1}{2}$ ,  $x \succeq y$  and  $y \succeq x$ , therefore  $x \sim y$ . Since  $y = \frac{1}{2} = z - \frac{1}{2}$ ,  $y \succeq z$  and  $z \succeq y$ , therefore  $y \sim z$ . However, we don't have  $x \succeq z$  because  $x = 0 < \frac{1}{2} = z - \frac{1}{2}$ . Therefore  $x \sim z$  is not satisfied.

(c)

If we had  $x, y \in X = \{0, 1, 2, \dots\}$ ,

$$x \succeq y \Leftrightarrow x \geq y - 1/2 \Leftrightarrow x \geq y.$$

Then both completeness and transitivity are satisfied, so  $\succeq$  is a preference relation.

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## Question 2

Since the agent is risk neutral, his von-Neuman Morgenstern utility function is represented by the amount of money he receives (or any affine transformation of it).

(a)

If he doesn't buy the security, the amount of money he has at date 1 is 0 for sure, so the expected utility is 0. If he buys the security at price  $p$ , the amount of money he has at date 1 is  $100 - p$ ,  $50 - p$ , or  $-p$ , each with probability  $1/3$ , and thus the expected utility is

$$\frac{1}{3}(100 - p) + \frac{1}{3}(50 - p) + \frac{1}{3}(-p).$$

He wants to buy this security if and only if this expected utility is higher than or equal to 0, the expected utility from not buying.

$$\frac{1}{3}(100 - p) + \frac{1}{3}(50 - p) + \frac{1}{3}(-p) \geq 0 \Leftrightarrow p \leq 50.$$

Therefore  $\pi_S = 50$ .

(b)

If the agent exercises the option, he will pay  $K$  and receive the dividend  $d$ , so he will receive net payment of  $d - K$ . If he doesn't exercise the option, he receives 0. Therefore he will exercise if and only if  $d - K \geq 0$ . His utility is represented by

$$u(d) = \max\{0, d - K\} - p,$$

where  $p$  is the price of the option.

(c)

**Case 1**  $K > 100$

In this case, the agent will never exercise the option, so he ends up with receiving 0 no matter what the dividend is. Therefore he wants to pay for this option no more than 0.

**Case 2**  $50 < K \leq 100$

If  $d = 100$ , the agent will exercise the option and receive net payment of  $100 - K$ . However, if  $d = 0$  or  $50$ , he won't exercise the option and receive 0. Therefore, he wants to buy this option if

$$\begin{aligned} \frac{1}{3}(100 - K - p) + \frac{2}{3}(-p) &\geq 0 \\ \Leftrightarrow p &\leq \frac{1}{3}(100 - K). \end{aligned}$$

**Case 3**  $0 \leq K \leq 50$

If  $d = 50$  or  $100$ , the agent will exercise the option and receive net payment of  $d - K$ . However, if  $d = 0$ , he won't exercise the option and receive  $0$ . Therefore, he wants to buy this option if

$$\begin{aligned} \frac{1}{3}(100 - K - p) + \frac{1}{3}(50 - K - p) + \frac{1}{3}(-p) &\geq 0 \\ \Leftrightarrow p &\leq 50 - \frac{2}{3}K. \end{aligned}$$

To summarise,

$$\pi_O = \begin{cases} 0 & \text{if } K > 100 \\ \frac{1}{3}(100 - K) & \text{if } 50 < K \leq 100 \\ 50 - \frac{2}{3}K & \text{if } 0 \leq K \leq 50 \end{cases}$$

**Question 3**

Here is an example.

	L	R
s	4,0	0,0
s'	0,0	6,0
s''	1,1	1,1

The mixed strategy  $\frac{1}{2}s + \frac{1}{2}s'$  gives player 1 expected utility of 2 if player 2 plays l and expected utility of 3 if player 2 plays r.

**Question 4**

The intuition tells us only 0 would be selected, but actually, all strategies except 100 are rationalizable! The reason is that a player's objective is just to win the game, i.e., to name closer number to one third of the average than everyone else, and not to name exactly one third of the average. For example, if everybody else is naming 100, you can win by naming any number below 100, even 99 wins. The proof is given in the answer 1.

But if, not like this game, a player's objective is to name exactly one third of the average, our first intuition works. The proof for this case is given in the answer 2.

**Answer 1**

If  $0 \leq x < 100$ ,  $x$  is the best response to all other players' playing  $x'$ , where  $x < x' < 100$  and  $x'$  is close enough to  $x$ . To see this, if a player plays  $x$  and all others plays  $x'$ ,  $\bar{x}/3 < x$  (because  $x'$  is close enough to  $x$ ), therefore she is the

only winner. Since she can do no better than being the only winner,  $x$  is the best response.

For any  $0 \leq x < 100$ , we can find a sequence  $x < x_1 < x_2 < \dots < 100$  such that  $x$  is a best response to everyone else's playing  $x_1$ ,  $x_1$  is a best response to everyone else's playing  $x_2$ , and so on.

Naming 100 is not a best response to anything. The only case a player wins is when everyone is naming 100, but even in this case, she can be the only winner by naming a smaller number.

To conclude,  $x$  is rationalizable if and only if  $0 \leq x < 100$ .

## Answer 2

Suppose the award is  $100 - |x - \frac{x_i}{3}|$ , so that a player always want to name exactly  $\frac{x_i}{3}$ .

Given that  $0 \leq x_i \leq 100$  for all  $i$ ,  $\frac{x_i}{3}$  always lies between 0 and  $33.33\dots$ . So  $x > 33.33\dots$  is never best response, i.e., it is not rationalizable.

As long as everyone is naming  $0 \leq x_i \leq 33.33\dots$ ,  $\frac{x_i}{3}$  always lies between 0 and  $11.11\dots$ . So  $x > 11.11\dots$  is never best response, i.e., it is not rationalizable.

Repeating this procedure forever, any number greater than 0 is not rationalizable.

If everyone else is naming 0, the best response is to name 0 as well. Therefore 0 is rationalizable.

To conclude,  $x = 0$  is the only rationalizable strategy.

Bonus: In the discussion above, the number of players played no role at all.

## Question 5

(a)

	l	r
$L\lambda\Lambda$	2,2	2,2
$L\lambda P$	2,2	2,2
$L\lambda u$	2,2	2,2
$L\rho\Lambda$	2,2	2,2
$L\rho P$	2,2	2,2
$L\rho u$	2,2	2,2
$R\lambda\Lambda$	1,3	3,3
$R\lambda P$	1,3	1,1
$R\lambda u$	1,3	0,0
$R\rho\Lambda$	3,1	3,3
$R\rho P$	3,1	1,1
$R\rho u$	3,1	0,0

(b)

First,  $R\rho\Lambda$  weakly dominates all other strategies of player 1. We eliminate them and then we have the following game.

	$l$	$r$
$R\rho\Lambda$	3,1	3,3

Now,  $l$  is weakly dominated by  $r$ , so we eliminate  $l$  and we have

	$r$
$R\rho\Lambda$	3,3