Bargaining Theory I

MIT 14.126 Game Theory

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Bargaining Theory

- **Cooperative (Axiomatic)**
  - **Edgeworth**
  - **Nash Bargaining (*)**
  - **Variations of Nash**
    - **Kalai-Smorodinsky**
    - **Maschler-Perles**
    - **Egalitarian-Equivalent**
    - **Utilitarian, etc.**
  - **Shapley Value (*)**
- Non-cooperative
  - Rubinstein-Stahl (*) (complete info)
  - Asymmetric info
    - Rubinstein, Admati-Perry, Crampton, …
  - Non-common priors
    - Posner, Bazerman, Yildiz (*), …

Nash Bargaining Problem

- $N = \{1,2\}$ – the agents
- $S \subset R^N$ -- the set of feasible expected-utility pairs
- $d = (d_1,d_2) \in S$ – the disagreement payoffs
- A *bargaining problem* is any (S,d) where
  - S is compact and convex, and
  - $\exists x \in S$ s.t. $x_1 > d_1$ and $x_2 > d_2$.
- B is the set of all bargaining problems.
- A *bargaining solution* is any function

  $f : B \to R^N$ s.t. $f(S,d) \in S$ for each (S,d).

Nash Axioms

1. **Expected-utility Axiom [EU]** (invariance under affine transformations): $\forall(S,d), \forall(S',d'), a_i > 0$

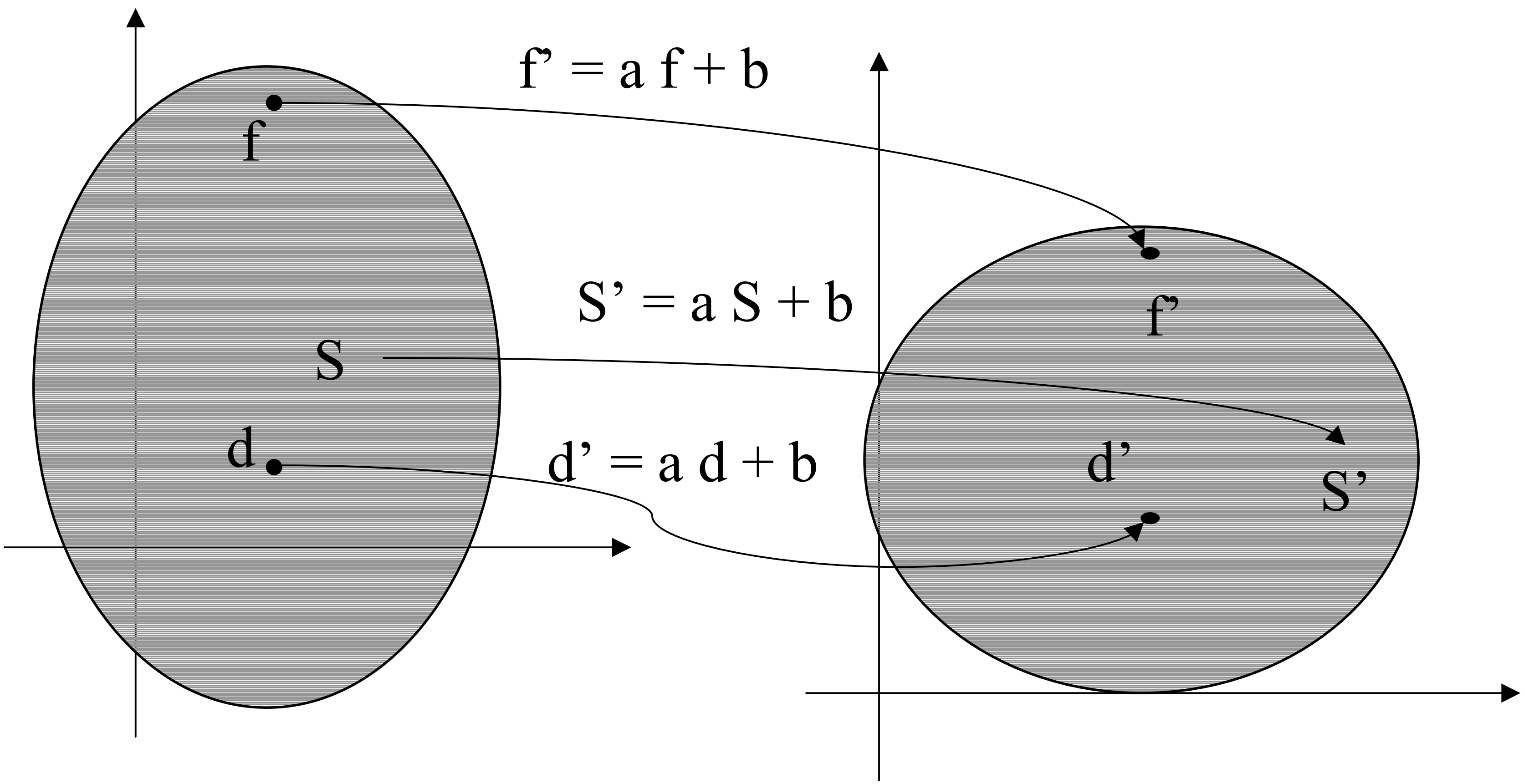
$$\left.\begin{array}{l} S' = \{s' \mid s'_i = a_i s_i + b_i \ \forall i \in N\} \\ d'_i = a_i d_i + b_i \ \forall i \in N \end{array}\right\} \Rightarrow f_i(S',d') = a_i f_i(S,d) + b_i \ \forall i \in N$$

2. **Symmetry [Sy]:** Let (S,d) be symmetric: $d_1 = d_2$ and $[(x_1,x_2) \in S$ iff $(x_2,x_1) \in S]$. Then,

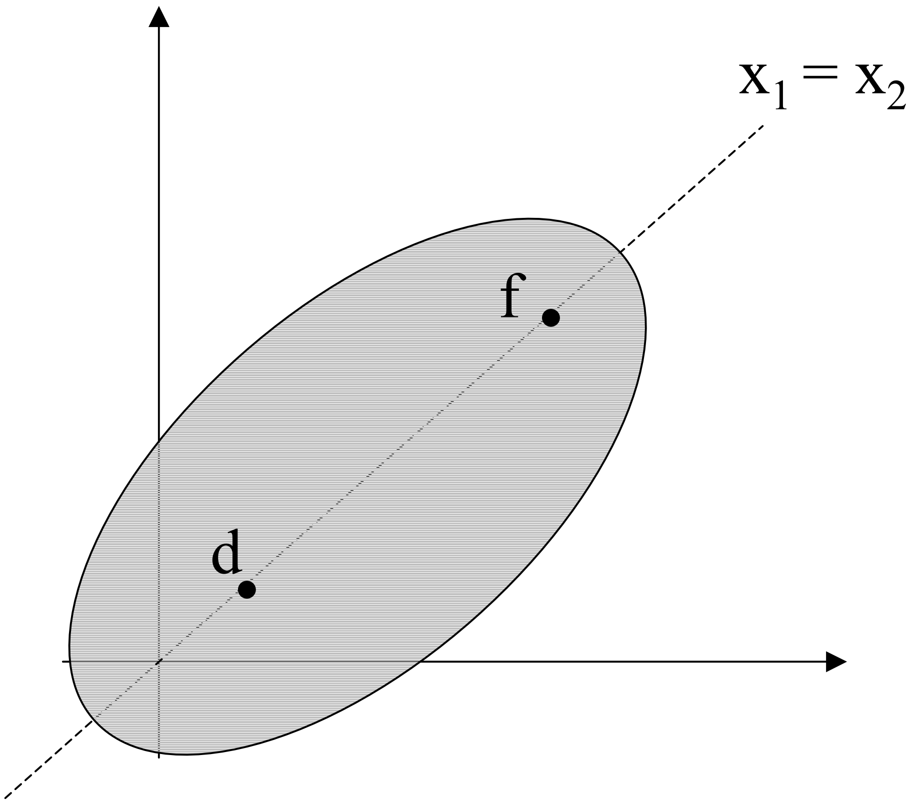
$$f_1(S,d) = f_2(S,d).$$

3. **Independence of Irrelevant alternatives [IIA]**: if $T \subset S$ and $f(S,d) \in T$, then $f(T,d) = f(S,d)$.
4. **Pareto – Optimality [PO]:** if $x,y \in S$ and $y > x$, then $f(S,d) \neq x$.

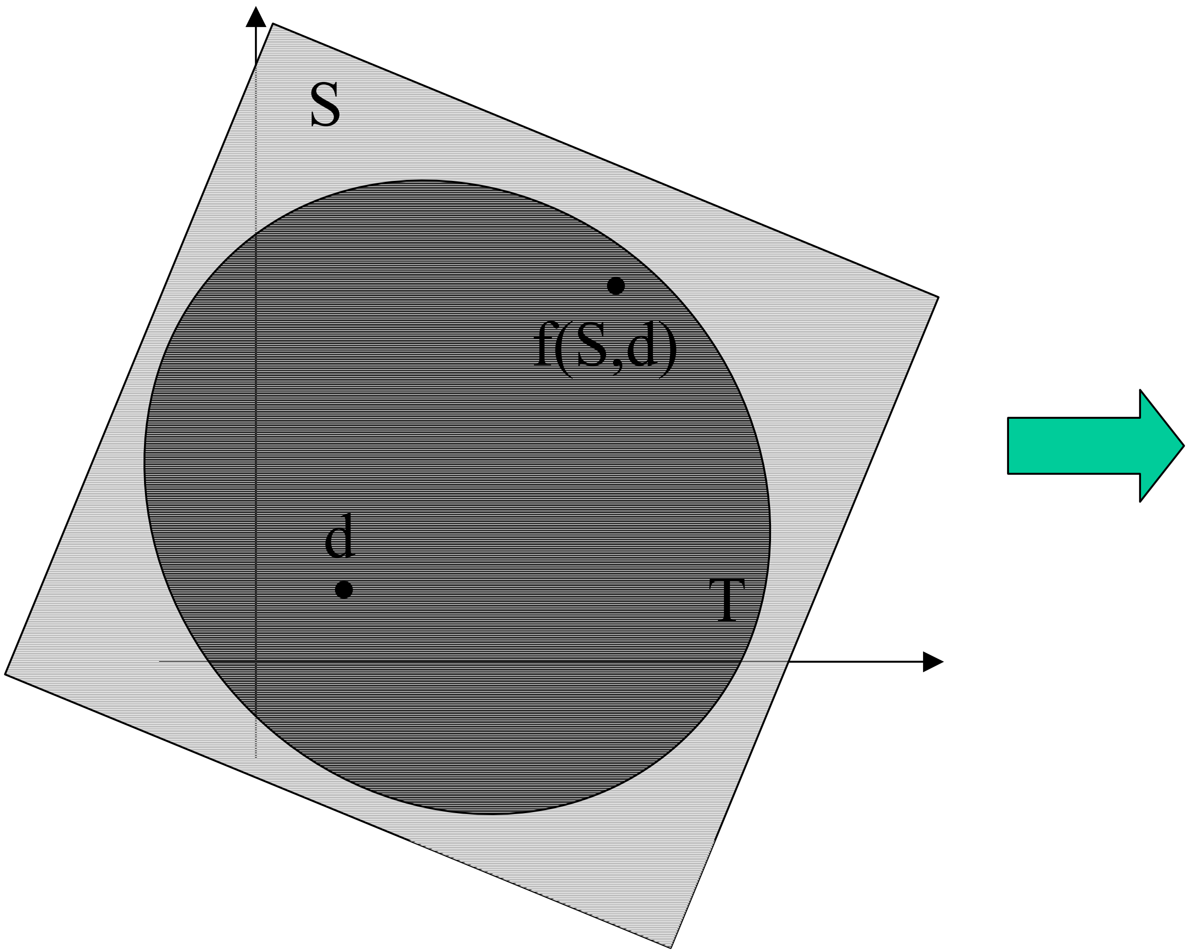
Expected-utility Axiom



Symmetry



Independence of irrelevant alternatives

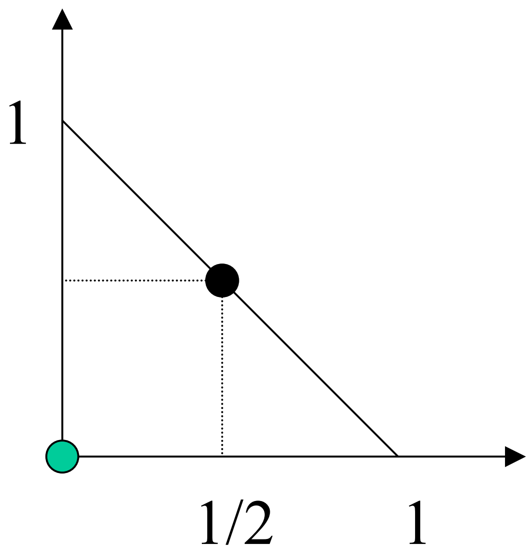


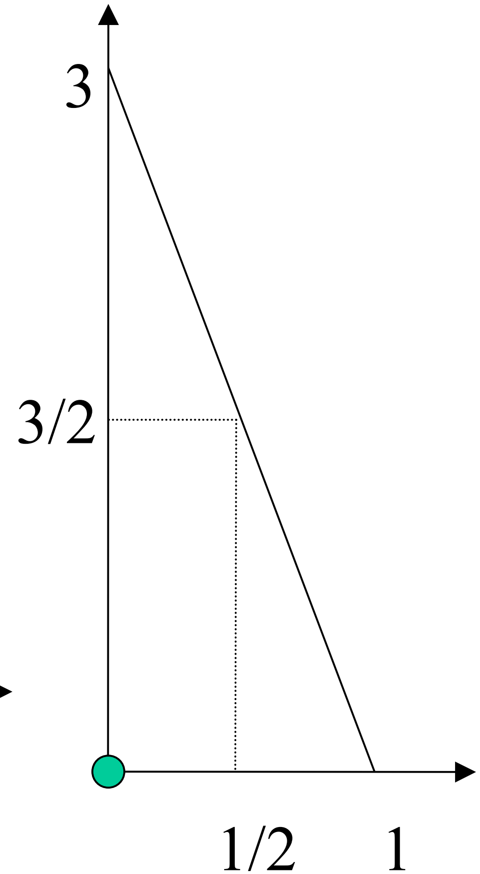
f(T,d) = f(S,d)

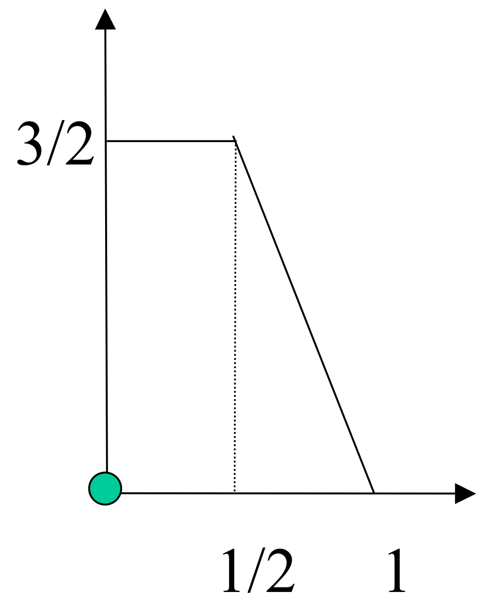
Nash Bargaining Solution

$$f^*(S,d) = \underset{\substack{s \equiv (s_1, s_2) \in S \\ s > d}}{\arg\max} (s_1 - d_1)(s_2 - d_2).$$

Examples







$$f^*(S,d) = \arg\max_{\substack{s \equiv (s_1, s_2) \in S \\ s > d}} (s_1 - d_1)(s_2 - d_2).$$

Nash's Theorem

Theorem: A bargaining solution *f* satisfies the Nash Axioms (EU,Sy,IIA,PO) if and only if

$$f = f^*.$$

Nash Axioms

1. **Expected-utility Axiom** (invariance under affine transformations): $\forall(S,d), \forall(S',d'), a_i > 0$

$$\left.\begin{aligned} S' &= \{s' \mid s'_i = a_i s_i + b_i \ \forall i \in N\} \\ d'_i &= a_i d_i + b_i \ \forall i \in N \end{aligned}\right\} \Rightarrow f_i(S',d') = a_i f_i(S,d) + b_i \ \forall i \in N$$

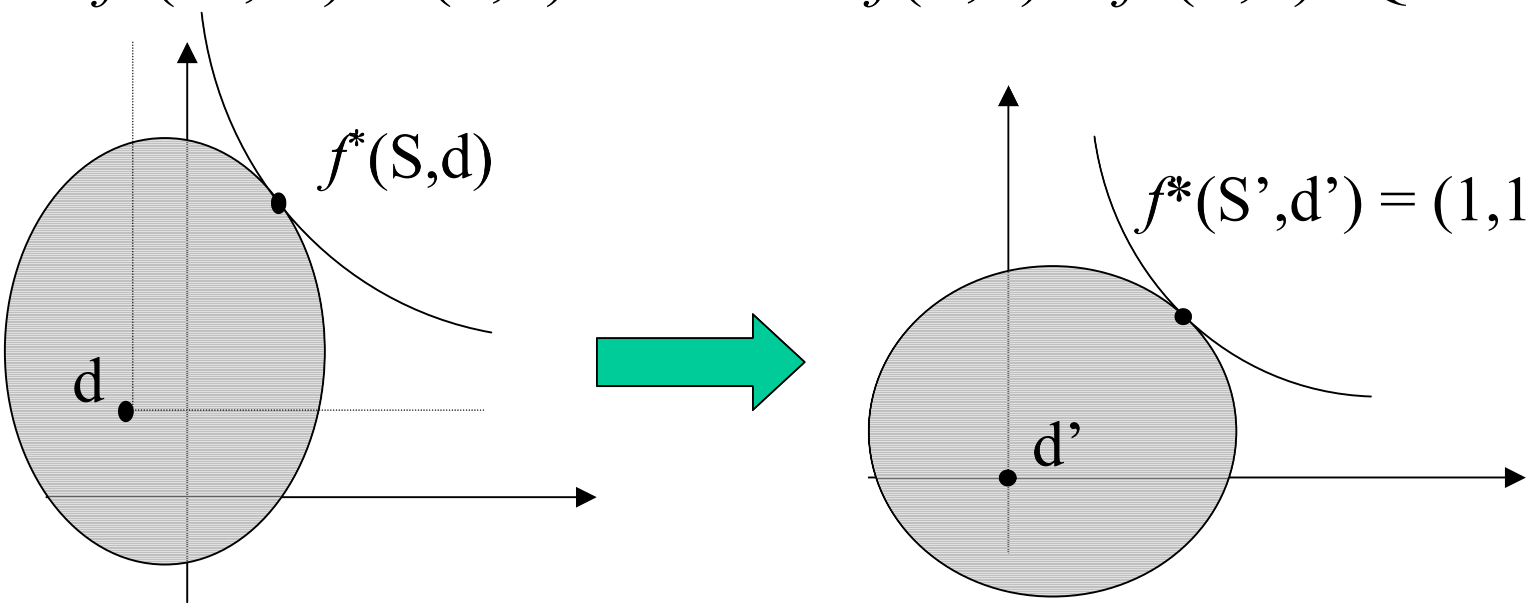
2. **Symmetry:** Let (S,d) be symmetric: $d_1 = d_2$ and $[(x_1,x_2) \in S \text{ iff } (x_2,x_1) \in S]$. Then,

$$f_1(S,d) = f_2(S,d).$$

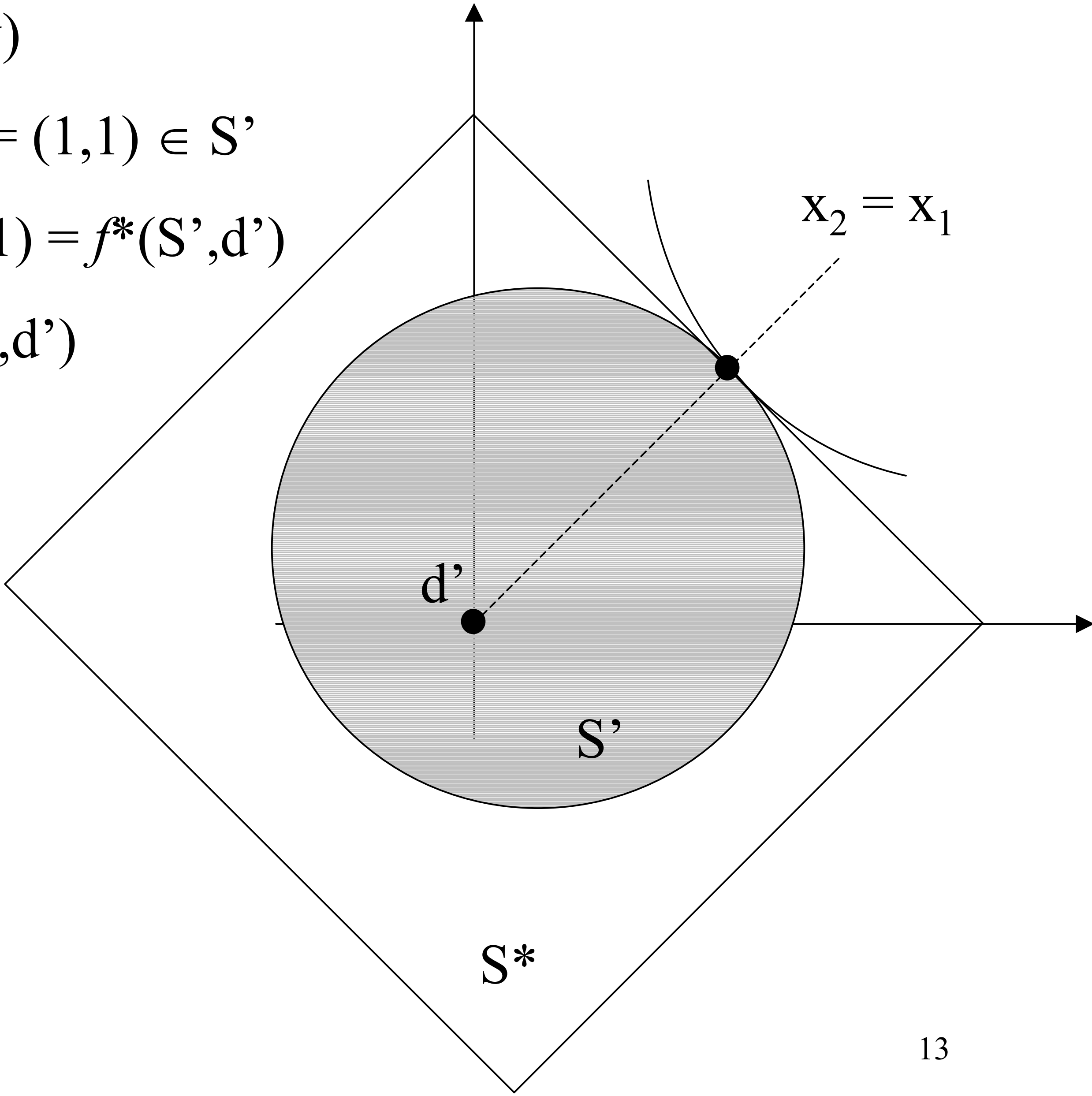
3. **Independence of Irrelevant alternatives**: if $T \subset S$ and $f(S,d) \in T$, then $f(T,d) = f(S,d)$.
4. **Pareto – Optimality:** if $x,y \in S$ and $y > x$, then $f(S,d) \neq x$.

Proof of Nash's Theorem

1. Check: f^* satisfies the Nash axioms. (easy)
2. Take any (S,d). Transform it to (S',d') so that d' = 0, and f^*(S',d') = (1,1). Under [Sy,IIA,PO], f(S',d') = f^*(S',d') = (1,1). &EU $\Rightarrow$ f(S,d) = f^*(S,d). QED



- S^* is symmetric. (how)
- &Sy&PO $\Rightarrow f(S^*,d') = (1,1) \in S'$
- &IIA $\Rightarrow f(S',d') = (1,1) = f^*(S',d')$
- &EU $\Rightarrow f(S,d) = f^*(S',d')$



An extension of Nash

5. **Individual Rationality [IR]:** f(S,d) $\geq$ d.

Theorem: There are precisely two bargaining solutions that satisfy axioms EU,Sy,IIA, and IR: *f** and D with D(.,d) $\equiv$ d.

Proof: [EU&IIA&IR] => (PO or D(.,d) $\equiv$ d). QED

Asymmetric Nash

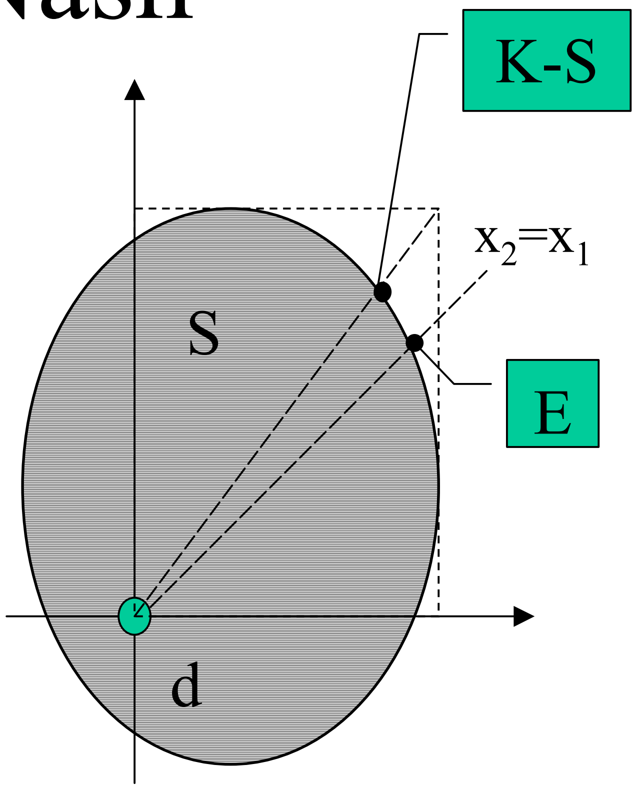
Theorem: Let A = $\{x \geq 0 | x_1 + x_2 \leq 1\}$. For any a in (0,1), there exists a unique b.s. f^a that satisfies Axioms EU,IIA, and IR, and $f^a(A,0) = (a,1-a)$;

$$f^a(S,d) = \arg\max_{s \in S, s \geq d} (s_1 - d_1)^a (s_2 - d_2)^{1-a}$$

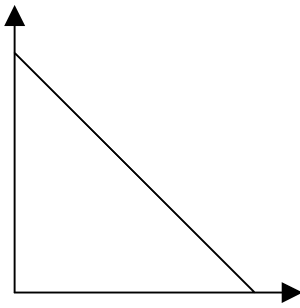
Variations of Nash

Changing the Nash's axioms, many characterized various b.s. with various axioms, e.g.,

1. Kalai-Smorodinsky
   - Monotonicity, EU, Sy, PO
2. Egalitarian: maxmin$\{x_1,x_2\}$
3. Utilitarian: max $a x_1+b x_2$



Shapley Value – n person bargaining



- A coalitional game (N,v), where $v : 2^N \rightarrow R$. v(S) is the maximum total utility the coalition S can get in the case of disagreement with N\S.
- A *bargaining solution* (or a *value*) is any function f that assigns an allocation f(S,v) in R^S for each coalition S, where $\Sigma_i f_i(S,v) = v(S)$.
- The *marginal contribution* of *i* to S with $i \notin S$ is

$$D_i(S) = v(S \cup \{i\}) - v(S).$$

Shapley Value -- definition

- A coalition $S_i = \{1,2,\ldots,i\}$
  - formed in the order $\{1\} \rightarrow \{1,2\} \rightarrow \{1,2,3\} \rightarrow \ldots \rightarrow \{1,2,\ldots,i\text{-}1\} \rightarrow \{1,2,\ldots,i\text{-}1,i\}$;
  - the new-comer has all the bargaining power.
- Then, $f_1(S_i,v) = v(\{1\})$, $f_2(S_i,v) = D_2(\{1\}) = v(\{1,2\}) - v(\{1\})$, … , $f_i(S_i,v) = D_i(S_{i-1}) = v(\{1,2\}) - v(\{1\})$.

- Coalition S
  - formed in a random order where each permutation is equally likely – there are |S|! Perms.;
  - the new-comer has all the bargaining power.
- Then, Shapley Value (φ):

$$\varphi_i(S,v) = \frac{1}{|S|!}\sum_R D_i(S_i(R))$$

where R is any permutation, $S_i(R) = \{R(1),R(2),\ldots,i\}$.

Example -- Firm

- $N = \{c\} \cup W$; c owns a factory; $w \in W$ is a worker. Without c, workers produce 0; with c, m workers produce p(m); p is concave, increasing, and p(0) = 0.
- $v(S) = p(|S \cap W|)$ if $c \in S$; v(S) = 0 otherwise.

[O&R;259.3]

- $\varphi(c) = \varphi(\omega) = 0$;
- $\varphi_c(\{c,w\}) = \varphi_w(\{c,w\}) = p(1)/2$;
- $\varphi_c(\{c,w_1,w_2\}) = \varphi_w(\{c,w_1,w_2\}) = p(2)/3$.
- …
- $\varphi_c(\{c,w_1,w_2,\ldots,w_m\}) = \varphi_w(\{c,w_1,w_2,\ldots,w_m\}) = p(m)/(m+1)$.

Example -- Market

- $N = \{1,2,3\}$; 1 is seller; 2, 3 are buyers:
- $v(i) = 0$; $v(1,2) = v(1,3) = v(1,2,3) = 1$; $v(2,3) = 0$.
- $\varphi_i(i) = 0$; $\varphi_1(1,i) = \varphi_i(1,i) = ½$; $\varphi_i(2,3) = 0$;

  $\varphi_1(1,2,3) = 2(0 + 1 + 1)/3! = 2/3$;

  $\varphi_2(1,2,3) = \varphi_3(1,2,3) = 1/3! = 1/6$.

[the price is 2/3, and buyers have equal probability of buying]

- $Core(N,v) = \{(1,0,0)\}$.

Shapley value & the Core

Theorem: For any convex game (N,v), the Shapley value (φ) is in the core.

Proof:

1. Since (N,v) is convex, ∀ perm. R, g^R with $g_i^R(N,v) = D_i(S_i(R))$ is in the Core (previous lecture).
2. Shapley value is the average of g^R's: $\varphi = \Sigma_R g^R/|N|!$
3. The Core is convex.
4. Shapley value is in the Core. QED

Shapley Axioms

1. **Symmetry:** If i and j are interchangeable (i.e., $D_i = D_j$), then
$$f_i(.,v) = f_j(.,v).$$
2. **Dummy:** If i is dummy (i.e., $D_i = v(\{i\})$), then
$$f_i(.,v) = v(\{i\}).$$
3. **Additivity**: $f(.,v+w) = f(.,v) + f(.,w)$.

Theorem (Shapley)

The Shapley value (φ) is the unique bargaining solution (or value) that satisfies the Shapley axioms (namely, symmetry, dummy, and additivity).

Proof:

1. Check: φ satisfies the Shapley axioms.
2. There exists a unique value f that satisfies the Shapley axioms. QED

Proof (Step 2)

1. Fix N. So, $(N, v) \equiv v \in R^{2^{|N|}-1}$.
2. Define v_T by $v_T(S) = 1$ if $S \subseteq T$; $v_T(S) = 0$ otherwise.
3. $(v_T)_{\emptyset \neq T \subseteq N}$ is a basis for $R^{2^{|N|}-1}$:
   1. Suppose $\Sigma_S b_S v_S = 0$, but $b_T \neq 0$.
   2. $\exists T^* \subseteq T$ s.t. $b_{T^*} \neq 0$ & $b_{T'} = 0$ $\forall T' \subset T^*$.
   3. $\Sigma_S b_S v_S(T^*) = b_{T^*} \neq 0$, a contradiction.
4. $\forall v \in R^{2^{|N|}-1}$, $\exists$ a unique $b \in R^{2^{|N|}-1}$ s.t. $v = \Sigma_S b_S v_S$.
5. A1&A2 => $f_i(a v_T) = a/|T|$ if $i \in T$; 0 otherwise.
6. &A3 => $f_i(v) = f_i(\Sigma_S b_S v_S) = \Sigma_S f_i(b_S v_S) = \Sigma_{S \ni i} b_S/|S|$.

Balanced Contributions

A value f satisfies the balanced contributions property iff $\forall$(N,v), $\forall$i,j in N,

$$f_i(N,v) - f_i(N\backslash\{i\},v) = f_j(N,v) - f_j(N\backslash\{j\},v).$$

Theorem: The Shapley value is the only bargaining solution that satisfies the balanced contributions property.

Proof: 1. If f and f' satisfy the property, then f = f'.
2. Shapley value satisfies the property. QED