# Game Theory is Evolving

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# Our Topics in the Course

- Classical Topics
  - Choice under uncertainty
  - Cooperative games
    - Values
    - 2-player bargaining
    - Core and related
    - Cores of market games
  - Non-cooperative games
    - Fixed points/equilibrium
    - Refinements
    - (Repeated games)
- Newer Topics
  - "Nash program" (noncoop foundations for coop games)
  - Cheap talk
  - Experiments
  - Foundations of Noncoop Games
    - Epistemic
    - Learning
    - Evolutionary
  - Mechanism design
  - Other economic applications
    - Commitment, signaling
    - Comparative statics



# Two Game Formulations

The cooperative concept
- Perfect communication
- Perfect contract enforcement

- Formulation (N,v)
  - N is a set of players
  - TU case: $v:2^N \rightarrow R$
  - NTU case: $v(S):2^N \rightarrow R^S$
- Solution(s): (set of) value allocations $\pi \in R^N$

The non-cooperative concept
- No communication
- No contract enforcement

- Formulation $(N,S,\pi)$
  - N is a set of players
  - $x \in S = S_1 \times \ldots \times S_N$ is a strategy profile
  - $\pi_n(x)$=n's payoff
- Solution(s): (set of) strategy profiles $x \in S$

# Example: Linear Duopoly

- Non-cooperative
  - Players: 1 and 2
  - Strategies: Output levels $x_1$ and $x_2$
  - Payoffs: $x_i[1-(x_1+x_2)-c_i]$
  - Nash Equilibrium:

    $x_1=(1+c_2-2c_1)/3$

    $x_2=(1+c_1-2c_2)/3$
- Cooperative TU

  $v(1)=v(2)=0?$

  $v(1,2)=\max_x x(1-x-\min(c_1,c_2))$
- Cooperative NTU

  $v(1)=v(2)=0?$

  $v(1,2)=\{(\pi_1,\pi_2):(\exists x_1,x_2\in[0,1])$

  $\pi_1 = x_1(1-x_1-x_2-c_1)$ &

  $\pi_2 = x_2(1-x_1-x_2-c_2)\}$
- Solution?

# Cooperative Solutions



# Solutions: Points & Sets

- Cooperative "one point" solutions usually attempt to extend a symmetric solution to asymmetric games.
  - "Nash bargaining solution"
  - "Shapley value"
  - In the TU Cournot game, the solution might be $\pi_j = v(i) + ½[v(1,2) - v(i)]$ with $v(i)=0$ or Cournot profit.
- Others generalize the indeterminacy of the bargaining outcome and identify only what is "blocked."
  - In the NTU Cournot game, the "core" consists of all the efficient, individually rational outcomes.



# The Shapley Value $\varphi_n(N,v)$

- Four axioms for TU games
  - Efficiency: $\sum \varphi_n(N,v)=v(N)$
  - Null player: If $v(S\cup\{n\})\equiv v(S)$ then $\varphi_n(N,v)=0$.
  - Symmetry: Permutations don't matter
  - Additivity: $\varphi(N,v+w)=\varphi(N,v)+\varphi(N,w)$
- Analysis:
  - The games (N,v) (with N fixed) form a linear space.
  - Let $\chi_S(T)=1$ if $S\subset T$ and $\chi_S(T)=0$ otherwise.
  - Axiom 1-3: $\varphi_n(\alpha\chi_S)=\alpha/|S|$ if $n\in S$ and $\varphi_n(\alpha\chi_S)=0$ if $n\notin S$.
  - Add Axiom 4: $\varphi$ is a linear operator

# Shapley's Theorem

- Notation:
  - $\Pi$ = set of permutations of N, typical element $\pi$, mapping elements of N onto $\{1,\ldots,|N|\}$.
  - $S_{i\pi}=\{j:\pi_j\leq\pi_i\}$
- <u>Theorem</u>.

$$\varphi_n(N,v)=\frac{1}{|N|!}\sum_{\pi\in\Pi} v(S_{i\pi})-v(S_{i\pi}\setminus i)$$

- <u>Proof</u>. The given $\varphi$ satisfies the axioms by inspection. Since the $\chi_S$ games form a basis, there is a unique linear operator that does so. QED

# Shapley Intepretations

- Power, for example in voting games.
- Fairness, for example in cost allocations.
- Extension: Aumann-Shapley pricing
  - Cornell long-distance phone cost allocation
  - Game has K types of players with mass $a_k$ of type k.
  - "Diagonal formula":

$$\varphi_k(N,v) = \int v_k(sa_1,\ldots,sa_K)ds$$



# Crossover Ideas

Connecting Cooperative and Non-cooperative game ideas



# Crossover Ideas, 1

- Cheap Talk
  - What happens to non-cooperative games when we add a stage of message exchange?
  - In the following game, suppose
    - Payoff unit is $10,000s
    - Row player can send a message…

| | |
|---|---|
| 7,6 | 8,5 |
| 0,0 | 9,9 |

# Crossover Ideas, 2

- The "Nash Program"
  - Non-cooperative foundations for cooperative solutions
  - 2-player bargaining problem
    - Nash's bargaining solution
    - Nash demand game
    - Stahl-Rubinstein "alternating offers" model
  - Exchange economies
    - Competitive equilibrium (core)
    - Shapley-Shubik game
    - Auction and bidding games

# Nash Demand Bargaining

- Let $v(1)=v(2)=0$, $v(12)=1$.
  - The "core" is the set of payoff vectors $\pi$ that are:
    - "Efficient": $\sum_{n\in N} \pi_n=v(N)$
    - "Unblocked": $(\forall S\subset N)\ \sum_{n\in S} \pi_n\geq v(S)$
  - In this case, $Core(N,v)=\{(\pi_1,\pi_2): \pi_1,\pi_2\geq 0, \pi_1+\pi_2=1\}$.
- Nash demand game. $S_1=S_2=[0,1]$.
  - $$\pi_n = \begin{cases} x_n & \text{if } x_1 + x_2 \leq 1 \\ 0 & \text{otherwise} \end{cases}$$
  - "Perfect" equilibria coincide with the core.



# Alternating Offer Bargaining

- Players 1 and 2 take turn making offers.
- If agreement $(x_1,x_2)$ is reach at time t, payoffs are $\delta^t(x_1,x_2)$.
- Unique subgame perfect equilibrium has payoffs of "almost" the (.5,.5) Nash solution:

$$\pi_n = \begin{cases} \frac{1}{1+\delta} & \text{if n moves first} \\ \frac{\delta}{1+\delta} & \text{otherwise} \end{cases}$$



# Noncooperative Theory

## Rethinking equilibrium

# What are we trying to model?

- Animal behavior:
  - evolutionary stable strategies
- Learned behavior:
  - Reinforcement learning models
  - Self-confirming equilibrium
- Self-enforcing agreements:
  - Nash equilibrium & refinements
- Reflection among rational players
  - "Interactive epistemology"

# Animal Behavior

- Hawk-Dove game: what behavior will evolve?

| | |
|---|---|
| -1,-1 | 2,0 |
| 0,2 | 1,1 |



# Human Experiments

- The "Ultimatum Game"
  - Players: {1,2}
  - Strategies
    - Player 1 makes an offer x,1-x; x∈[0,1]
    - Player 2 sees it and says "yes" or "no"
  - Payoffs
    - If 2 says "yes," payoffs are (x,1-x).
    - If 2 says "no," payoffs are (0,0).
- Analysis
  - Unique subgame perfect equilibrium outcome: (x=1,yes).
  - It never happens that way!

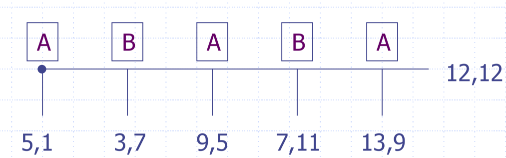


# Learning Models

- Behavior is established by some kind of adaptive learning
  - Kind #1: Players repeat strategies that were "successful" in "similar" past situations
  - Kind #2: Players forecast based on others' past play and, eventually, optimize accordingly.
    - Stationary points: "fulfilled expectations" equilibrium

# Epistemic Analyses

- "What I do depends on what I expect."
- The "centipede game"
  - Is it rational below for A to grab $5?
  - What should B believe if its probability one forecast of A's behavior is contradicted?



# Forward Induction

- Are both subgame perfect equilibria reasonable as rational play?