Replicator dynamics & Evolutionary stability

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"After all, insects can hardly be said to think at all, an so rationality cannot be so crucial if game theory somehow manages to predict their behavior under appropriate conditions. Simultaneously the advent of experimental economics brought home the fact that human subjects are no great shakes at thinking either. When they find their way to an equilibrium of a game, they typically do so using trial-and-error methods."

Ken Binmore

1 Common models of Learning or/and Evolution

1. Fictitious play.

$$k_0^i = S^{-i} \to \mathbb{R}_+;$$
$$k_{t+1}^i(s^{-i}) = \begin{cases} k_t^i(s^{-i}) + 1, \text{ if } s^{-i} = s_t^{-i}, \\ k_t^i(s^{-i}), \text{ if } s^{-i} \neq s_t^{-i}; \end{cases}$$
$$\gamma_t^i(s^{-i}) = \frac{k_t^i(s^{-i})}{\sum_{s^{-i}} k_t^i(s^{-i})},$$
$$\text{FP}: \ \rho^i(\gamma_t^i) \in BR^i(\gamma_t^i).$$

2. Partial best-response dynamics.

3. Replicator dynamics.

4. Evolutionary stable strategies.

5. Stochastic adjustment models.

2 Convergence, steady states

State variable: θ_t (discrete or continuous). Law of motion:

$$\begin{aligned} \theta_{t+1} &= f_t(\theta_t), \;\; \Pr(\theta_{t+1} = \theta) = \phi_t(\theta|\theta_t); \\ \dot{\theta}_t &= f_t(\theta_t). \end{aligned}$$

Flow of the system: $\theta_t = F_t(\theta_0)$, $F_{t+1}(\theta_0) = f_t(F_t(\theta_0))$, $D_t(F_t(\theta_0)) = f_t(F_t(\theta_0))$.

Steady State: $\hat{\theta}$, $F_t(\hat{\theta}) = \hat{\theta}$, $t > 0$.

SS is *stable*: for any U of $\hat{\theta}$ exists U_1 of $\hat{\theta}$ such that if $\theta_0 \in U$, $F_t(\theta_0) \in U_1$, $\forall t > 0$.

SS is *asymptotically stable:* if it is stable and

$$\lim_{t\to\infty} F_t(\theta_0) = \hat{\theta}.$$

Define: Basin of attraction, global stability, local isolation, hyperbolic *SS* (sink, sourse, saddle).

3 Evolutionary Stable Strategies

3.1 Notation

- $G = (S, A)$ a symmetric (for now), 2-player game where

- S is the strategy space;

- $A_{i,j} = u_1(s_i, s_j) = u_2(s_j, s_i)$;

- $x, y \in \Delta$ are mixed strategies; $u(x, y) = x^T Ay$;

- $u(ax + (1 - a)y, z) = au(x, z) + (1 - a)u(y, z)$.

3.2 Evolution stability

- Each player is endowed with a strategy (population/mutant strategy).

- Does not explain how a population arrives at such a strategy.

- Ask whether a strategy is robust to evolutionary pressures.

- Disregards effects on future actions.

3.3 ESS

Definition: A (mixed) strategy x is said to be *evolutionarily stable* iff, given any $y \neq x$, there exists $\epsilon_y > 0$ s.t.

$$u(x, (1-\varepsilon)x + \varepsilon y) > u(y, (1-\varepsilon)x + \varepsilon y),$$

for each ε in $(0, \epsilon_y]$.

Fact: x is evolutionarily stable iff, $\forall y \neq x$,

1. $u(x, x) \geq u(y, x)$, and
2. $u(x, x) = u(y, x) \Longrightarrow u(x, y) > u(y, y)$.

Proof: Define *score function*

$$\begin{aligned}
F_x(\varepsilon, y) &= u(x, (1-\varepsilon)x + \varepsilon y) - u(y, (1-\varepsilon)x + \varepsilon y) \\
&= u(x-y, x) + \varepsilon u(x-y, y-x). \\
\text{ESS} &\iff F_x(\varepsilon, y) > 0 \text{ for } \varepsilon \in (0, \epsilon_y].
\end{aligned}$$

3.4 ESS vs NE

- If $x \in \Delta^{ESS}$ then (x,x) is NE ($x \in \Delta^{NE}$).

  In fact: (x,x) is proper NE.

- (x,x) is strict NE $\Longrightarrow$ x is ESS by default.

- Interior NE may not be ESS.

- Suppose $x \in \Delta^{ESS}$, $C(y) \subset C(x)$, then $y \notin \Delta^{NE}$.

3.5 Hawk-Dove game

		
	$\left(\frac{V-c}{2}, \frac{V-c}{2}\right)$	$(V,0)$
	$(0,V)$	$(V/2,V/2)$

Example: For $V = 4$, $c = 6$; $x = \left(\frac{2}{3}, \frac{1}{3}\right)$—NE; $\forall y \in \Delta$, $y \in BR(x)$.

$$u(x - y, y) = (x_1 - y_1)(2 - 3y_1) = \frac{1}{3}(2 - 3y_1)^2,$$

so, x is ESS.

3.6 Rock-Scissors-Paper game

	R	S	P
R	0,0	1,-1	-1,1
S	-1,1	0,0	1,-1
P	1,-1	-1,1	0,0

- Unique Nash Equilibrium (s^*, s^*), where $s^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

- s^* is not ESS. $(u(s^* - R, R) = 0)$.

3.7 ESS and cheap talk

A coordination game:

	L	R
L	2,2	-100,0
R	0,-100	1,1

A secret handshake story: (RR) and no message is not ESS.

Surprisingly (?): In the unique ESS no information is being transmitted.

3.8 ESS in role-playing games

- Given (S^1, S^2, u_1, u_2), consider symmetric game (S, u), where
  - $S = S^1 \times S^2$;
  - for $x = (x_1, x_2), y = (y_1, y_2) \in S$

$$u(x, y) = \frac{1}{2}[u_1(x_1, y_2) + u_2(x_2, y_1)].$$

Theorem: x is ESS of (S, u) iff x is a strict NE of (S^1, S^2, u_1, u_2).

4 Replicator dynamics (homog. P)

- Selection mechanism.

- $p_i(t) = \#$people who plays s_i at t.

- $p(t) =$ total population at t.

- $x_i(t) = \frac{p_i(t)}{p(t)}$; $x(t) = (x_1(t), \ldots x_k(t))$.

- $u(x, x) = \sum_i x_i u(s_i, x)$.

- *Replicators* are pure strategies

$$\dot{p}_i = [\beta + u(s_i, x) - \delta] \, p_i.$$

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$$\dot{x}_i = [u(s_i, x) - u(x, x)] \, x_i = u(s_i - x, x) x_i.$$

4.1 Observations

- Suboptimal strategies can increase their shares (unlike FP or BR dynamics)

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$$\frac{d}{dt}\left[\frac{x_i}{x_j}\right] = \frac{\dot{x}_i}{x_j} - \frac{x_i}{x_j}\frac{\dot{x}_j}{x_j}$$
$$= \left[u(s_i, x) - u(s_j, x)\right]\frac{x_i}{x_j}.$$

- If u becomes $u' = au + b$, then Replicator dynamics becomes

$$\dot{x}_i = au(s_i - x, x)x_i.$$

4.2 2×2 games

Consider (S, A), where $A = \begin{bmatrix} a_1, a_1 & 0, 0 \\ 0, 0 & a_2, a_2 \end{bmatrix}$.

We have

$$\begin{aligned} u(s_i, x) &= a_i x_i; \\ u(x, x) &= (x_1, x_2) A (x_1, x_2)^T = a_1 x_1^2 + a_1 x_2^2; \\ u(s_1 - x, x) &= (a_1 x_1 - a_2 x_2) x_2. \end{aligned}$$

and so

$$\dot{x}_1 = (a_1 x_1 - a_2 x_2) x_1 x_2.$$

4.3 Classification

1. $a_1 a_2 < 0$. Then

   - $x_1 \rightarrow_t 0$ when $a_1 < 0$;

   - $x_1 \rightarrow_t 1$ when $a_1 > 0$.

2. $a_1 a_2 > 0$; define $\lambda = \frac{a_2}{a_1+a_2}$, $(\lambda, 1-\lambda)$ is NE. Then,

   - $x_1 = \lambda$ is stable if $a_1 < 0$;

   - $x_1 = \lambda$ is unstable if $a_1 > 0$.

Compare with ESS.

Examples: Prisoner's dilemma, Chicken, Coordination game, Battle of the sexes, ...

4.4 RD in Rock-Scissors-Paper game

	R	S	P
R	1,1	2+a,0	0,2+a
S	0,2+a	1,1	2+a,0
P	2+a,0	0,2+a	1,1

- Unique Nash Equilibrium (s^*, s^*), where $x^* = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

- Define $h(x) = \log(x_1 x_2 x_3)$.

- $$\dot{h}(x) = \frac{a}{2}\left(3\,\|x\|^2 - 1\right).$$

- $\min_x \|x\| = 1$, $\arg\min = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

4.5 Dynamics

Three scenarios:

1. $a = 0$ — original RSP; all trajectories are cycles.

2. $a < 0$ — x^* is unstable.

3. $a > 0$ — x^* is stable.

4.6 Rationalizability

- $\boldsymbol{\xi}(t, x_0)$ is the solution to replicator dynamics starting at x_0.

Theorem: If a pure strategy i is strictly dominated (by y), then $\lim_t \xi_i(t, x_0) = 0$ for any interior x_0.

Proof: Define $v_i(x) = \log(x_i) - \sum_j y_j \log(x_j)$. Then,

$$\begin{aligned}
\frac{dv_i(x(t))}{dt} &= \frac{\dot{x}_i}{x_i} - \sum_j y_j \frac{\dot{x}_j}{x_j} \\
&= u(s_i - x, x) - \sum_j y_j u(s_j - x, x) \\
&= u(s_i - y, x) \leq -\epsilon < 0.
\end{aligned}$$

Hence, $v_i(\boldsymbol{\xi}(t, x_0)) \to -\infty$, so $\xi_i(t, x_0) \to 0$.

Theorem: If i is not rationalizable, then $\lim_t \xi_i(t, x_0) = 0$ for any interior x_0.

4.7 Theorems

Theorem: Every ESS x is an asymptotically stable steady state of replicator dynamics.

(If the individuals can inherit the mixed strategies, the converse is also true.)

Proof: Define $C = supp(x)$, $Q = \{y | C \subset supp(y)\}$, $H(y) = \sum_{i \in C} x_i \log(y_i)$.

1. x is a local maximum of H, and

2. $\exists$ a neighborhood $n(x)$ s.t. H is increasing along any trajectory in $Q \cap n(x)$.

$$\dot{H} = \sum_{i \in C} x_i \frac{\dot{y}_i}{y_i} = \sum_{i \in C} x_i u(s_i - y, y) = u(x - y, y) > 0.$$

NE $\longrightarrow$ Steady state in RD;

Stable SS in RD $\longrightarrow$ NE.

Theorem: If x is an asymptotically stable steady state of replicator dynamics, then (x, x) is a perfect Nash equilibrium.

Proof:

1. (x, x) is a Nash equilibrium.
   (a) x is stable $=>$ $\dot{x}_i = u(s_i - x, x)x_i = 0$.
   (b) Suppose $(x, x) \notin NE$.
   (c) $\exists i \notin supp(x)$: $u(s_i - x, x) > 0$. [by 1 and 2]
   (d) $\exists \delta > 0$, $n(x)$: $u(s_i - y, y) > \delta \ \forall y \in n(x)$.
   (e) $\xi_i(t, y^0) > y_i^0 e^{\delta t}$ if $\xi_i(\cdot, y^0)$ remained in $n(x)$.
2. x is not weakly dominated (since ASS).

4.8 Non-ESS asymptotic stability

	L	M	R
L	0,0	1,-2	1,1
M	-2,1	0,0	4,1
R	1,1	1,4	0,0

- $NE = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$; mutant $= \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

- RD is asymptotically stable.

- Note: If mixed strategies can be inherited, $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ becomes instable.

4.9 General dynamics

Definition: A process is *payoff monotone* iff, at each interior x,

$$u(s_i, x) > u(s_j, x) \Leftrightarrow \frac{\dot{x}_i}{x_i} > \frac{\dot{x}_j}{x_j}.$$

Theorem: Under any any “regular” payoff monotone dynamics, if strategy i is eliminated by the process of iterated pure strategy strict dominance, then $\lim_t x_i(t) = 0$.

4.10 Social learning

- Ask around; if the other person does better, adopt his strategy.

Emulation dynamics ("medium-enhancing"):

Player 2 is a dummy, $p(L) = \frac{1}{3}$.

	L	R
U	9,0	0,0
D	2,0	2,0

- Ask around; if the other makes u' and you make u, then switch with probability max$\{0, b(u\prime - u)\}$.

- Aspiration levels.

4.11 Stimulus-response

- $u(x, y) \in [0, 1]$

- $x_i^k(t+1) = (1-\gamma u(s^k(t), \cdot))x_i^k(t) + F(s^k(t), i)\gamma u(s^k(t), \cdot),$
  where

$F(s^k(t), i) = 1$ if $s^k(t) = i$,

$F(s^k(t), i) = 0$ otherwise.

- Result: As γ goes to 0, trajectories converge to the RD trajectories.