Auctions 2:

Departures from symmetric IPV

Multi-unit auctions

1 IPV and Revenue Equivalence: Key assumptions

- Independence of values.
- Risk-neutrality.
- No budget constraints.
- Symmetry (Same allocation rule!).
- Other considerations:
  - Collusion
  - Resale possibilities

2 Risk-averse bidders

- Each bidder has $u : \mathbb{R}_+ \to \mathbb{R}$ with $u(0) = 0$, $u' > 0$, and $u'' < 0$.

Proposition: With risk-averse symmetric bidders the expected revenue in a first-price auction is greater than in a second-price auction.

Intuition: Consider a bidder in the first-price auction.

By reducing current bid b by some Δ, a bidder gains Δ when wins, but increases a probability of losing, which has a greater effect on expected utility.

Result: more aggressive bidding in the first-price auction.

No change in strategies in the second-price auction.

Formally, suppose $\gamma : [0, w] \to \mathbb{R}_+$ is an equilibrium strategy (incr. diff.).

$$\max_z EU = \max_z G(z)u(x - \gamma(z)).$$

FOC:

$$g(z)u(x - \gamma(z)) - G(z)u'(x - \gamma(z))\gamma'(z) = 0.$$

In symmetric eqm:

$$\begin{aligned} \gamma'(z) &= \frac{u(x - \gamma(z))}{u'(x - \gamma(z))} \frac{g(x)}{G(x)}, \\ \beta'(z) &= (x - \beta(x)) \frac{g(x)}{G(x)}. \end{aligned}$$

Note that for all $y > 0$, $\frac{u(y)}{u'(y)} > y$. Therefore,

$$\gamma'(z) > (x - \gamma(x)) \frac{g(x)}{G(x)}.$$

Now $\beta(x) > \gamma(x) \Rightarrow \gamma'(x) > \beta'(x)$. Together with $\beta(0) = \gamma(0) = 0$ we obtain

$$\beta(x) < \gamma(x).$$

3 Budget-constrained bidders

- Every bidder obtains value (signal) $X_i \in [0,1]$ and absolute budget $W_i \in [0,1]$.

- (X_i, W_i) are iid across bidders. (X_i and W_i need not be independent.)

Proposition: With budget-constrained bidders the expected revenue in a first-price auction is greater than in a second-price auction. (provided symmetric equilibrium exists.)

Intuition: The bids in second-price auction are higher on average and so are more often constrained.

(Not enough: players will reduce bids in the first-price auction).

Proof: In the second-price auction:

$$\beta^{\mathrm{II}}(x,w) = \min\{x,w\}.$$

Define (effective type) $x^{\mathrm{II}} \sim (x,w)$ as the type that is effectively unconstrained and submits the same bid as (x,w). Can be found as a solution to

$$\beta^{\mathrm{II}}(x,w) = \beta^{\mathrm{II}}(x^{\mathrm{II}},1) = x^{\mathrm{II}}.$$

Let $Y_2^{\mathrm{II}(N)}$ be the second highest of the equivalent values, x_i^{II}, among N bidders. Its distribution is

$$G^{\mathrm{II}}(z) = \left(F^{\mathrm{II}}(z)\right)^{N-1},$$

where $F^{\mathrm{II}}(z)$ is the probability that $\beta^{\mathrm{II}}(x,w) = \beta^{\mathrm{II}}(x^{\mathrm{II}},1) = x^{\mathrm{II}} < z = \beta^{\mathrm{II}}(z,1)$.

We have

$$E[R^{\mathrm{II}}] = E\left[Y_2^{\mathrm{II}(N)}\right].$$

In the first-price auction: Suppose a symmetric increasing equilibrium exists with

$$\beta^{\mathrm{I}}(x,w) = \min\{\beta(x),w\}.$$

Define $x^{\mathsf{I}} \sim (x, w)$ as the solution to

$$\beta^{\mathsf{I}}(x, w) = \beta^{\mathsf{I}}(x^{\mathsf{I}}, 1) = \beta(x^{\mathsf{I}}) < x^{\mathsf{I}}.$$

Let $Y_2^{\mathsf{I}(N)}$ be the second highest of the equivalent values, x_i^{I}, among N bidders. Its distribution is

$$G^{\mathsf{I}}(z) = \left(F^{\mathsf{I}}(z)\right)^{N-1}.$$

We have

$$E[R^{\mathsf{I}}] = E\left[Y_2^{\mathsf{I}(N)}\right].$$

Note that $F^{\mathsf{I}}(z) < F^{\mathsf{II}}(z)$, and thus

$$E[R^{\mathsf{I}}] > E[R^{\mathsf{II}}].$$

All-pay auctions dominate first-price auctions in terms of revenue generated to the seller.

4 Asymmetric bidders

- Revenue equivalence theorem applies only to mechanisms (equilibria) with the same allocation rule.

- Second price auction is efficient.

- First price auction generally is not.

  - Weaker bidder will bid higher.

- No general revenue ranking.

5 Resale (and efficiency)

- Intuition: If resale is possible, low-value bidders will bid more aggressively: revenue to the seller should be higher.

  Counter-argument: High-value bidders will bid less, and possibly will not reveal their values via bidding in the first period.

- If outcome is efficient after resale (no additional information is exogenously revealed) revenue equivalence holds.

- Second-price auction with resale: efficient, no resale happens.

  In general: Any efficient mechanism followed by resale would have an equilibrium like that.

- First-price auction with resale: inefficient in general (asymmetry), values are not revealed in the first period.

A simple illustration: Two bidders, $F_1[0,w] \neq F_2[0,w]$, $E[X_1] \neq E[X_2]$.

The winner can make a take it or leave it offer to the loser.

Claim: There is no efficient equilibrium in the first-price auction followed by resale that reveals valuations of the bidders in the first stage. (Why then efficiency would not be possible to obtain in general?)

Suppose β_1 and β_2 are increasing strategies with inverses $\phi_i = \beta_i^{-1}$.

Step 1. $\beta_1(w) = \beta_2(w) = \bar{b}$.

Step 2. Use revenue equivalence. Expected payments of a bidder with X_i have to be the same here and in the second price auction. Thus, $\beta_1(w) = E[X_2] \neq E[X_1] = \beta_2(w)$.

6 Collusion

Very brief:

- Typically modelled as bidding rings.

A bidding ring is a collection of bidders who exchange information, decide on the participation in the auction (who and how bids), decide on transfers.

Analysis: Stability of a ring (coalition),

Effects on the other bidders, and the seller.

Counter-measures by the seller.

- Second-price auction.

A group of bidders exchange information (conduct an auction among themselves), the winner goes to the main auction and bids her value, others do not go or bid 0.

The ring obtains (in case of win)

$$\max\left\{Y_1^{\mathcal{N}\setminus i}, r\right\} - \max\left\{Y_1^{\mathcal{N}\setminus \mathcal{I}}, r\right\},$$

where i is the winner and $\mathcal{I}$ is the ring.

Relatively easy to support. No bidder from the ring can go (incognito) to the main auction and benefit (need to overbid).

Seller might respond by setting a higher reserve price.

- First-price auction.

The same structure roughly.

Now, however, a “representative” bidder can send a “friend” who will just overbid him, and thus capture all the spoils without sharing them among the members of the ring.

Other types of collusion:

Seller can cheat by inserting “fake” bids — has an effect in the second-price auction.

In case of multiple units, by specific bidding patterns buyers can signal their intentions and support collusion.

- Practical auction design: entry and collusion.

  Better to attract another bidder and have no reserve than to set an optimal reserve price. (Symmetry is crucial).

  English auction vs sealed-bid auctions: discourages entry, more susceptible to collusion. Openness and information revelation (feedback) maybe crucial.

  Multiple units for sale: other issues, parallel simultaneous or sequential ascending price auctions are particularly susceptible.

7 Multi-unit auctions

M units of the same object are offered for sale.

Each bidder has a set of (marginal values) $V^i = (V^i_1, V^i_2, \dots V^i_M)$, the objects are substitutes, $V^i_k \geq V^i_{k+1}$.

Extreme cases: unit-demand, the same value for all objects.

- Types of auctions:
- The discriminatory ("pay-your-bid");
- Uniform-price;
- Vickrey;
- Multi-unit English;
- Ausubel;
- Dutch, descending uniform-price,
- ...

Issues: Existence and description of equilibria, price series if sequential, efficiency, optimality, non-homogenous goods, complementarities,...

8 Interdependent (common) values.

- Each bidder receives private signal $X_i \in [0, w_i]$. ($w_i = \infty$ is possible)

- $(X_1, X_2, \ldots, X_n)$ are jointly distributed according to commonly known F ($f > 0$).

- $$V_i = v_i(X_1, X_2, \ldots, X_n).$$
  $$v_i(x_1, x_2, \ldots, x_n) \equiv E\left[V_i \mid X_j = x_j \text{ for all } j\right].$$
  Typically assumed that functional forms $\{v_i\}_{i=1}^N$ are commonly known.

- $v_i(0, 0, \ldots, 0) = 0$ and $E[V_i] < \infty$.

- Symmetric case:
  $$v_i(x_i, \mathbf{x}_{-i}) = v(x_i, \mathbf{x}_{-i}) = v(x_i, \pi(\mathbf{x}_{-i})).$$

9 Brief analysis

- Common values / Private values / Affiliated values / Interdependent values.

- Winner's curse.

- Second-price auction: Pivotal bidding—I bid what I get if i just marginally win.

- First-price auction: "Usual" analysis—differential equation, ...

- English auction: See below.

- Revenue ranking: English > SPA > FPA.

  (!) Interdependency and affiliation are important for the first part.

10 Second-price auction

Define

$$v(x,y) = E\left[V_1 \mid X_1 = x,\ Y_1 = y\right].$$

Equilibrium strategy

$$\beta^{\mathrm{II}}(x) = v(x,x).$$

Indeed,

$$\begin{aligned}\Pi(b,x) &= \int_0^{\beta^{-1}(b)} (v(x,y) - \beta(y))\, g(y|x)dy \\ &= \int_0^{\beta^{-1}(b)} (v(x,y) - v(y,y))\, g(y|x)dy.\end{aligned}$$

Π is maximized by choosing $\beta^{-1}(b) = x$, that is, $b = \beta(x)$.

11 Example

1. Suppose S_1, S_2, and T are uniformly and independently distributed on $[0,1]$. There are two bidders, $X_i = S_i + T$. The object has a common value
$$V = \frac{1}{2}(X_1 + X_2).$$

2. In this example, in the first price auction:
$$\beta^{\mathrm{I}}(x) = \frac{2}{3}x, \qquad E[R^I] = \frac{7}{9}.$$

3. In the second-price auction $v(x,y) = \frac{1}{2}(x+y)$ and so
$$\beta^{\mathrm{II}}(x) = x, \qquad E[R^I] = \frac{5}{6}.$$

12 Linkage principle

Define

$$W^A(z,x) = E\left[P(z) \mid X_1 = x,\ Y_1 < z\right]$$

expected price paid by the winning bidder when she receive signal x but bids z.

Proposition: (Linkage principle):

Let A and B be two auction forms in which the highest bidder wins and (she only) pays positive amount. Suppose that symmetric and increasing equilibrium exists in both forms. Suppose also that

1. for all x, $W_2^A(x,x) \geq W_2^B(x,x)$.
2. $W^A(0,0) = W^B(0,0) = 0$.

Then, the expected revenue in A is at least as large as the expected revenue in B.

So,the greater the linkage between a bidder's own information and how he perceives the others will bid the greater is the expected price paid upon winning.