Auctions 3:

Multi-unit auctions

Interdependent values

1 Multi-unit auctions

M units of the same object are offered for sale.

Each bidder has a set of (marginal values) $\mathbf{V}^i = (V_1^i, V_2^i, \ldots V_M^i)$, the objects are substitutes, $V_k^i \geq V_{k+1}^i$.

Special cases: unit-demand, the same value for all objects, reordered independent draws.

Types of auctions:

- The discriminatory ("pay-your-bid");
- Uniform-price;
- Vickrey;

- Multi-unit English;
- Ausubel;
- Dutch, descending uniform-price,
- ...

Issues: Existence and description of equilibria, price series if sequential, efficiency, optimality, non-homogenous goods, complementarities,...

1.1 Efficiency

Bids $\mathbf{b}^i = (b^i_1, \dots, b^i_M)$. Suppose objects are awarded to the M highest bids. Suppose also $f(\mathbf{V}^i) > 0$.

Bidding strategy: $\boldsymbol{\beta}^i : \mathbf{V}^i \to \mathbb{R}^M_+$, $\beta^i_k(\mathbf{V}^i) \geq \beta^i_{k+1}(\mathbf{V}^i)$.

Efficiency requires that the rankings of bids and values agree:

$$\forall i, j, k, l, \qquad V^i_k > V^j_l \Leftrightarrow \beta^i_k(\mathbf{V}^i) > \beta^j_l(\mathbf{V}^j).$$

Two implications:

- Separability: bids are unaffected by values of other objects;
- Symmetry across bidders and objects.

Can each bidder be treated as a player with M multiple identities each having a unit demand?

Are Vickrey, discriminatory, uniform-price auctions efficient?

Single-unit demand?

Discriminatory auctions: 2 bidders, 2 goods.

- High bids on both objects are identical.
$$\max_{\mathbf{V}^1} \beta_1(\mathbf{V}^1) = \bar{b} = \max_{\mathbf{V}^2} \beta_2(\mathbf{V}^2).$$
- Define $H_k(c) = \Pr\left[\beta_k(\mathbf{V}) \le c\right].$
-
$$\begin{aligned} \Pi(\mathbf{b}, \mathbf{V}) &= H_1(b_2)(V_1 + V_2 - b_1 - b_2) \\ &\quad + \left[H_2(b_1) - H_1(b_2)\right](V_1 - b_1) \\ &= H_2(b_1)(V_1 - b_1) + H_1(b_2)(V_2 - b_2). \end{aligned}$$
- Remember: $b_1 \ge b_2$, therefore the problem above separates in two if $b_1 > b_2$ and is joint if $b_1 = b_2$.
- There must be a range where $b_1 = b_2$. (Asymmetric bidders analysis).

2 Interdependent (common) values.

- Each bidder receives private signal $X_i \in [0, w_i]$. ($w_i = \infty$ is possible)

- $(X_1, X_2, \ldots, X_n)$ are jointly distributed according to commonly known F ($f > 0$).

- $$V_i = v_i(X_1, X_2, \ldots, X_n).$$

  $$v_i(x_1, x_2, \ldots, x_n) \equiv E\left[V_i \mid X_j = x_j \text{ for all } j\right].$$

  Typically assumed that functional forms $\{v_i\}_{i=1}^N$ are commonly known.

- $v_i(0, 0, \ldots, 0) = 0$ and $E[V_i] < \infty$.

- Symmetric case:

  $$v_i(x_i, \mathbf{x}_{-i}) = v(x_i, \mathbf{x}_{-i}) = v(x_i, \pi(\mathbf{x}_{-i})).$$

- Affiliation: for all $\mathbf{x}', \mathbf{x}'' \in \mathcal{X}$,

  $$f(\mathbf{x}' \vee \mathbf{x}'')f(\mathbf{x}' \wedge \mathbf{x}'') \geq f(\mathbf{x}')f(\mathbf{x}'').$$

  If f is twice continuously differentiable and strictly positive, affiliation is equivalent to

  $$\frac{\partial^2 \ln f}{\partial x_i \partial x_j} \geq 0,$$

  also equivalent to require that $\ln f$ is a supermodular function (or that f is log-supermodular).

Examples:

- Oil field: each firm runs a test, obtains a noisy signal of capacity, ζ_i. The estimate of the expected value is $V_i = \frac{1}{N}\sum \zeta_i$, or, $V_i = a\zeta_i + (1-a)\frac{1}{N-1}\sum_{j \neq i} \zeta_j$; (not all firms may run tests, ..., other components to value); $V_i = X_i + \frac{1}{N}\sum \zeta_i$.

- Art painting: experts, dealers, collectors. Some bidders are more informative than the others.
- Spectrum licenses: Each of the incumbents knows its own set of customers, interested in knowing characteristics of the rest.
- Job market?

Note the difference: The actual 'value' impact of information vs strategic effect of knowing the other's values.

3 Brief analysis

- Common values / Private values / Affiliated values / Interdependent values.
- Winner's curse.
- Second-price auction: Pivotal bidding—I bid what I get if I just marginally win.
- First-price auction: "Usual" analysis—differential equation, ...
- English auction: See below.
- Revenue ranking: English $>$ SPA $>$ FPA.

  (!) Interdependency and affiliation are important for the first part.

4 Second-price auction

Define

$$v(x,y) = E\left[V_1 \mid X_1 = x,\ Y_1 = y\right].$$

Equilibrium strategy

$$\beta^{\mathsf{II}}(x) = v(x,x).$$

Indeed,

$$\begin{aligned}\Pi(b,x) &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - \beta(y)\right) g(y|x)dy \\ &= \int_0^{\beta^{-1}(b)} \left(v(x,y) - v(y,y)\right) g(y|x)dy.\end{aligned}$$

Π is maximized by choosing $\beta^{-1}(b) = x$, that is, $b = \beta(x)$.

5 Example

1. Suppose S_1, S_2, and T are uniformly and independently distributed on $[0,1]$. There are two bidders, $X_i = S_i + T$. The object has a common value
$$V = \frac{1}{2}\left(X_1 + X_2\right).$$

2. In this example, in the first price auction:
$$\beta^{\mathsf{I}}(x) = \frac{2}{3}x, \qquad E[R^I] = \frac{7}{9}.$$

3. In the second-price auction $v(x,y) = \frac{1}{2}(x+y)$ and so
$$\beta^{\mathsf{II}}(x) = x, \qquad E[R^I] = \frac{5}{6}.$$

6 Linkage principle

Define

$$W^A(z,x) = E\left[P(z) \mid X_1 = x,\ Y_1 < z\right]$$

expected price paid by the winning bidder when she receive signal x but bids z.

Proposition: (Linkage principle):

Let A and B be two auction forms in which the highest bidder wins and (she only) pays positive amount. Suppose that symmetric and increasing equilibrium exists in both forms. Suppose also that

1. for all x, $W_2^A(x,x) \geq W_2^B(x,x)$.
2. $W^A(0,0) = W^B(0,0) = 0$.

Then, the expected revenue in A is at least as large as the expected revenue in B.

So,the greater the linkage between a bidder's own information and how he perceives the others will bid the greater is the expected price paid upon winning.

7 Mechanism Design with non-independ valuations

7.1 Notation

K objects; given $\mathbf{k} = (k_1, \ldots, k_N)$, denote

$$\mathbf{V}^{\mathbf{k}} = \left(V_1^{k_1}, \ldots, V_N^{k_N}\right).$$

Winners circle at s, $\mathcal{I}^{\mathbf{k}}(\mathrm{s})$, is the set of bidders with the highest value among $\mathbf{V}^{\mathbf{k}}$.

$\mathbf{k}$ is *admissible* if $1 \leq k_i \leq K$ and

$$0 \leq \sum_{i=1}^{N}(k_i - 1) < K.$$

7.2 Single-crossing condition

MSC (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{x}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{x})$,

$$\frac{\partial V_i^{k_i}(\mathbf{x})}{\partial x_i} > \frac{\partial V_j^{k_j}(\mathbf{x})}{\partial x_i}.$$

7.3 Efficiency: VCG mechanism (generalized Vickrey auction)

- Allocation rule: Efficient.

- Payments: Vickrey price that player j pays for kth unit won:

$$\begin{aligned} p_j^k &= V_j^k(s_j^k, x_{-j}) = \\ & (M-k+1)\text{th highest} \\ & \text{among } \{V_i^m(s_j^k, x_{-j})\}_{i \neq j}^{m=1..M}. \end{aligned}$$

These are generically different across units and winners (unlike with private values).

7.4 Optimal Mechanisms

Non-independent values: Cremer-McLean mechanism extracts full surplus (and is efficient).

Discrete support: $\mathcal{X}^i = \{0, \Delta, 2\Delta, \ldots, (t_i - 1)\Delta\}$, discrete single-crossing is assumed (no need if the values are private).

$\Pi(\mathbf{x})$ is the joint probability of x, $\Pi_i = (\pi(\mathbf{x}_{-i}|x_i))$.

Theorem: In the above conditions and if Π has a full rank, there exists a mechanism in which truth-telling is an efficient ex post equilibrium and in which the seller extracts full surplus from the bidders.

Proof: Consider VCG mechanism $(\mathbf{Q}^*, \mathbf{M}^*)$. Define,

$$U_i^*(x_i) = \sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) \left[Q_i^*(\mathbf{x}) V_i(\mathbf{x}) - M_i^*(\mathbf{x})\right].$$

This is the expected surplus of buyer i in VCG mechanism. Define, $\mathbf{u}_i^* = (U_i^*(x_i))_{x_i \in \mathcal{X}^i}$.

There exists $\mathbf{c}_i = (c_i(\mathbf{x}_{-i}))_{\mathbf{x}_{-i} \in \mathcal{X}_{-i}}$, such that $\Pi_i \mathbf{c}_i = \mathbf{u}_i^*$. Equivalently,

$$\sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) c_i(\mathbf{x}_{-i}) = U_i^*(x_i).$$

Then, *CM* mechanism $(\mathbf{Q}^*, \mathbf{M}^{CM})$ is defined by

$$M_i^{\mathbf{CM}}(\mathbf{x}) = M_i^*(\mathbf{x}) + c_i(\mathbf{x}_{-i}).$$

Remarks:

- Private values (correlated), equiv. second price auction with additional payments.
- Negative payoffs sometimes, not ex post IR, payoffs arbitrarily large if the distribution converges to the independent one.