Auctions 4:

Multi-unit auctions

Interdependent values

(continued)

1 Efficient Mechanism Design

1.1 Notation

K objects; N bidders; each bidder receives a signal $s_i \in [0, 1]$.

$V_j^m(s_1, ..., s_n)$–marginal value of mth object to bidder j, $V_j^{m+1}(\mathbf{s}) \leq V_j^m(\mathbf{s})$.

Given $\mathbf{k} = (k_1, \ldots, k_N)$, denote

$$\mathbf{V}^{\mathbf{k}} = \left(V_1^{k_1}, \ldots, V_N^{k_N}\right).$$

This is a selection of marginal valuations of bidders according to $\mathbf{k}$.

Winners circle at $\mathbf{s}$, $\mathcal{I}^{\mathbf{k}}(\mathbf{s})$, is the set of bidders with the highest value among $\mathbf{V}^{\mathbf{k}}$.

$\mathbf{k}$ is *admissible* if $1 \leq k_i \leq K$ and

$$0 \leq \sum_{i=1}^{N}(k_i - 1) < K.$$

1.2 Single-crossing condition

MSC (*single-crossing*) For any admissible $\mathbf{k}$, for all $\mathbf{x}$ and any pair of players $\{i, j\} \subset \mathcal{I}^{\mathbf{k}}(\mathbf{x})$,

$$\frac{\partial V_i^{k_i}(\mathbf{x})}{\partial x_i} > \frac{\partial V_j^{k_j}(\mathbf{x})}{\partial x_i}.$$

2 VCG mechanism (generalized Vickrey auction)

Requires single-crossing.

- Allocation rule: Efficient. K highest marginal values win.

- Payments: Vickrey price that player j pays for kth unit won:

$$\begin{aligned} p_j^k &= V_j^k(s_j^k, x_{-j}) = \\ &\quad (M - k + 1)\text{th highest} \\ &\quad \text{among } \{V_i^m(s_j^k, x_{-j})\}_{i \neq j}^{m=1..M}. \end{aligned}$$

These are generically different across units and winners (unlike with private values).

- Can be used for sale of non-identical units, objects with complements, to price and find efficient allocation of various other problems.

3 Optimal Mechanisms

Non-independent signals: Cremer-McLean mechanism extracts full surplus (and is efficient).

Discrete support: $\mathcal{X}^i = \{0, \Delta, 2\Delta, \ldots, (t_i - 1)\Delta\}$, discrete single-crossing is assumed (no need if the values are private).

$\Pi(\mathbf{x})$ is the joint probability of x, $\Pi_i = (\pi(\mathbf{x}_{-i}|x_i))$.

Theorem: In the above conditions and if Π has a full rank, there exists a mechanism in which truth-telling is an efficient ex post equilibrium and in which the seller extracts full surplus from the bidders.

Proof: Consider VCG mechanism $(\mathbf{Q}^*, \mathbf{M}^*)$. Define,

$$U_i^*(x_i) = \sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) \left[Q_i^*(\mathbf{x}) V_i(\mathbf{x}) - M_i^*(\mathbf{x}) \right].$$

This is the expected surplus of buyer i in VCG mechanism. Define, $\mathbf{u}_i^* = (U_i^*(x_i))_{x_i \in \mathcal{X}^i}$.

There exists $\mathbf{c}_i = (c_i(\mathbf{x}_{-i}))_{\mathbf{x}_{-i} \in \mathcal{X}_{-i}}$, such that $\Pi_i \mathbf{c}_i = \mathbf{u}_i^*$. Equivalently,

$$\sum_{\mathbf{x}_{-i}} \pi(\mathbf{x}_{-i}|x_i) c_i(\mathbf{x}_{-i}) = U_i^*(x_i).$$

Then, *CM* mechanism $(\mathbf{Q}^*, \mathbf{M}^{CM})$ is defined by

$$M_i^{\mathbf{CM}}(\mathbf{x}) = M_i^*(\mathbf{x}) + c_i(\mathbf{x}_{-i}).$$

Remarks:

- Private values (correlated), equiv. second price auction with additional payments.
- Negative payoffs sometimes, not ex post IR, payoffs arbitrarily large if the distribution converges to the independent one.

4 English auction

- One object, asymmetric interdependent values, efficiency?

  Yes, if the number of bidders is 2 and single-crossing is satisfied.

- Not, if the number of bidders is 3 (plus SC). Need more.

- Yes if Generalized single-crossing is satisfied: arbitrarily increase signals of the subset of bidders from a winners' circle, a bidder from the subset must have the highest value.

- Yes, if modeled as an auction with reentry under even weaker conditions.

5 Multiple objects, indirect efficient mechanisms.

Main issue: winners and prices (Vickrey) are determined by different "thought" processes.

- Perry and Reny's ascending price auction;

- Perry and Reny's two-stage sealed bid auction;

- Dasgupta and Maskin's contingent bid auction;

- Sequential mechanism.