14.126
Fall 2003
Problem Set 2
Suggested Solutions

Question 1.

Let us calculate the payoff of player i with discount factor δ'' and continuation values $w''(y)$, for a given pure action a_i and α_{-i} strategy profile of the opponents:

$$\begin{aligned}
p_i(a_i, \alpha_{-i}, \delta'') &= (1-\delta'')g_i(a_i,\alpha_{-i}) + \delta'' \sum \pi_y(a_i,\alpha_{-i})w_i''(y) = \\
&= (1-\delta'')g_i(a_i,\alpha_{-i}) + \frac{\delta''-\delta'}{1-\delta'}v_i + \delta'\frac{1-\delta''}{1-\delta'}\sum \pi_y(a_i,\alpha_{-i})w_i'(y) = \\
&= \frac{\delta''-\delta'}{1-\delta'}v_i + \frac{1-\delta''}{1-\delta'}\left\{(1-\delta')g_i(a_i,\alpha_{-i}) + \delta'\sum \pi_y(a_i,\alpha_{-i})w_i'(y)\right\} = \\
&= \frac{\delta''-\delta'}{1-\delta'}v_i + \frac{1-\delta''}{1-\delta'}p_i(a_i,\alpha_{-i},\delta').
\end{aligned}$$

Thus the overall payoff with discount factor δ'' and continuation payoffs $w''(.)$ is a fixed monotone increasing (in fact linear) transform of the corresponding payoff with discount factor δ' and continuation payoffs $w'(.)$ for all opponent strategies α_{-i} and actions a_i. It follows that the optimal action choice under the new scenario is the same as it was under the old scenario.

Also, by assumption $p_i(a_i,\alpha_{-i},\delta') = v_i$ and therefore we can write

$$\begin{aligned}
p_i(a_i,\alpha_{-i},\delta'') &= \frac{\delta''-\delta'}{1-\delta'}v_i + \frac{1-\delta''}{1-\delta'}p_i(a_i,\alpha_{-i},\delta') = \\
&= \frac{\delta''-\delta'}{1-\delta'}v_i + \frac{1-\delta''}{1-\delta'}v_i = v_i.
\end{aligned}$$

Thus the optimal payoff under the new discount factor δ'' and continuation payoffs $w''(.)$ is the same as the optimal payoff under the old discount factor δ' and continuation payoffs $w'(.)$.

The only remaining thing to check is whether the vector w'' is in the required half-space for all y. Since $w''(y)$ is a convex combination of v and $w'(y)$, both of which are in the required half-space which is a convex set, this condition is also satisfied.

We have showed that under the new scenario the same action choice is optimal as before, yielding the same discounted payoff; we have also shown that the new continuation values are in the desired half-space. The proof is complete.

Question 2.

(a) Let $p_1 = Pr(a_1 =\text{Up})$, $p_2 = Pr(a_2 =\text{Up})$ and $q = Pr(a_3 =\text{Right})$. Since the choice of Heads or Tails does not matter for payoffs, we can calculate, given a triple (p_1, p_2, q), the expected payoff of player 3 to be

$$\pi(q, p_1, p_2) = -qp_1p_2 - (1-q)(1-p_1)(1-p_2).$$

The minmax payoff of player 3 is

$$\min_{p_1,p_2}\max_{q}\{-qp_1p_2-(1-q)(1-p_1)(1-p_2)\}$$

or, after reorganizing

$$-\max_{p_1,p_2}\min\{p_1p_2,\ (1-p_1)(1-p_2)\}.$$

Since

$$\begin{aligned}\min\{p_1p_2,\ (1-p_1)(1-p_2)\} &\le \sqrt{p_1p_2(1-p_1)(1-p_2)} = \\ &= \sqrt{p_1(1-p_1)}\sqrt{p_2(1-p_2)} \le \\ &\le \frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}\end{aligned}$$

it follows that

$$-\max_{p_1,p_2}\min\{p_1p_2,\ (1-p_1)(1-p_2)\} \ge -\frac{1}{4}.$$

Hence the minmax payoff of player 3 is at least $-1/4$. If both player 1 and 2 are mixing $1/2-1/2$ on Up and Down (that is, $p_1=p_2=1/2$) then all the inequalities above hold with equality, and the minmax value of $-1/4$ is achieved. Player 3 is indifferent between all possible strategies, so $\pi(q,1/2,1/2)=-1/4$ for all q.

(b) Following the example, the equilibrium we construct is the following:

- Player 1: Randomize $1/2-1/2$ between H and T in every period. If $y_{t-1}=1$ and player 2 played H, play D; If $y_{t-1}=1$ and player 2 played T, play U; If $y_{t-1}=0$ or $y_{t-1}=2$, randomize $1/2-1/2$ between U and D.
- Player 2: Randomize $1/2-1/2$ between H and T in every period. If $y_{t-1}=1$ and player 1 played H, play U; If $y_{t-1}=1$ and player 1 played T, play D; If $y_{t-1}=0$ or $y_{t-1}=2$, randomize $1/2-1/2$ between U and D.
- Player 3: Randomize $1/2-1/2$ between L and R.

It is easy to verify that this is a Nash equilibrium. Players 1 and 2 are indifferent between all possible outcomes. Given the constructed strategies of players 1 and 2, it is easy to see that player 3 is indifferent between playing R and playing L for any y_{t-1}, thus any strategy of player 3 is a best response. In particular so is the $1/2-1/2$ mix.

In all periods but the first one, player 3's payoff is as follows: she gets $-1/4$ with probability $1/2$ (when $y_{t-1}=0$ or $y_{t-1}=2$), and $-1/2$ with probability $1/2$ (when $y_{t-1}=1$, in which case players 1 and 2 coordinate on either (U,U) or (D,D), without player 3 knowing which one of the two is going to be played). Overall, the expected payoff of player 3 is $-3/8$.

In the very first period, there is no history, so player 3 obtains her minmax payoff of $-1/4$. Thus for any discount factor δ for player 3, her normalized payoff from the repeated game is

$$-(1-\delta)\frac{1}{4} - \delta\frac{3}{8} = -\frac{1}{4} - \frac{\delta}{8}$$

which is less than the minmax for any $\delta > 0$.

The reason that we obtain equilibrium payoffs that are lower than player 3's minmax value is that players 1 and 2 have an informational advantage. They use the coin flip as a common randomization device, unknown to player 3, which allows them to coordinate better on outcomes that make player 3 worse off. In the stage game this coordination was not possible. In particular, in the repeated game, for $y_{t-1} = 1$ players 1 and 2 could achieve a mix of (UU) and (DD) only, whereas in the non repeated game their optimal mix involved (UD) and (DU) as well.

Note that the strategies used by players 1 and 2 here are not public, and that the equilibrium found is not stationary. See also the discussion in FT page 188, preceding section 5.5.4.

Question 3.

(a) This part of the question asks us to find payoffs $g_i(a_i, x)$ for $a_i \in \{C, D\}$ and $x \in \{x', x''\}$ such that players' expected payoffs are as given in the normal form. Notice by symmetry we just need to show that we can generate the payoffs for player 1 in this way: payoffs for player 2 can be generated similarly. Now, generating the appropriate payoffs for player 1 is equivalent to solving the following set of linear equations:

$$\begin{pmatrix} 1 \\ -h \\ 1+d \\ 0 \end{pmatrix} = \begin{pmatrix} \pi_{x'}(C,C) & 1-\pi_{x'}(C,C) & 0 & 0 \\ \pi_{x'}(C,D) & 1-\pi_{x'}(C,D) & 0 & 0 \\ 0 & 0 & \pi_{x'}(D,C) & 1-\pi_{x'}(D,C) \\ 0 & 0 & \pi_{x'}(D,D) & 1-\pi_{x'}(D,D) \end{pmatrix} \cdot \begin{pmatrix} g(C,x') \\ g(C,x'') \\ g(D,x') \\ g(D,x'') \end{pmatrix}$$

This system will have a solution if the square matrix is invertible, which boils down to requiring that each of the two blocks on the diagonal are invertible. And this is immediate: the determinant of the upper left block, for example, is

$$\pi_{x'}(C,C)[1-\pi_{x'}(C,D)] - \pi_{x'}(C,D)[1-\pi_{x'}(C,C)]$$

which is clearly nonzero since $\pi_{x'}(C,D) > \pi_{x'}(C,C)$ and $1-\pi_{x'}(C,C) > 1-\pi_{x'}(C,D)$.

(b) To answer the later parts of the question, it's necessary to use the techniques of Fudenberg and Levine (1994) rather than the 'simpler' version in Fudenberg, Levine and Maskin (1994) section 3. In particular, to calculate the set $\lim_{\delta\to 1} E(\delta)$, we need to use the result that $E(\delta) = \cap_\lambda H^*(\lambda)$. Arguing simply that $E(\delta) \subset H^*((1,1))$ actually suffices for part (b) but definitely not for (c). The argument is due to Kandori and Obara (2003).

From individual rationality, we know that $E(\delta)$ is a subset of the first quadrant of $\mathbb{R}^2$ since the static Nash equilibrium payoffs $(0,0)$ provide a lower bound for each player's payoffs in any PPE. To establish the result in the question, it's sufficient to find the maximum score $k^*(\lambda)$ for λ with both λ_1 and λ_2 positive. Note that because of the asymmetry of the game, Lemma 4.1 of FL(1994) does not apply so that it does not suffice to work out the maximum score that can be enforced using the pure strategy profiles $\alpha = (C,C)$, $\alpha = (C,D)$ and $\alpha = (D,D)$ - we need to consider more general mixed strategies.

However, note that to obtain the point $(1,1)$ on the frontier, the players need to play C with sufficiently high probability and hence each player's defection makes x' more likely since $\pi_{x'}(C,C) < \pi_{x'}(C,D)$. Hence the players need to be punished when x' occurs. But this entails welfare loss, so it's impossible to sustain $(1,1)$. We now formalize this intuitive argument.

Choose λ so that $\lambda \cdot (1,1) \geq \lambda \cdot g(\alpha)$ for all α (for example, $\lambda = (1,1)$ will do). Now, $(1,1) \notin H(\lambda)$ is equivalent to

$$\lambda \cdot (1,1) > \sup \lambda \cdot ((1-\delta)g(\alpha) + \delta E_x[w(x) \mid \alpha])$$

where the supremum is taken over α and x such that α is enforced by w with $\lambda \cdot w(x) \leq \lambda \cdot (1,1)$ for all x. Here $g(\cdot)$ denotes expected payoffs today and $E_x[w(x) \mid \alpha]$ the expectation (taken over $x \in \{x', x''\}$) of promised values tomorrow).

Denote by D_ϵ the strategy of playing D with probability ϵ (and C with $1-\epsilon$). Let ϵ'' denote the minimum ϵ such that $\pi(x'|D_\epsilon, D) \leq \pi(X|D_\epsilon, C)$, and let $M = \{\alpha \mid \alpha_i(D) \geq \epsilon'', \, i = 1 \text{ or } 2\}$. Note if $\alpha \in M$ we have

$$\lambda \cdot (1,1) > \max_{\alpha \in M} (1-\delta)\lambda \cdot g(\alpha) + \delta\lambda \cdot (1,1) \geq \lambda \cdot ((1-\delta)g(\alpha) + \delta E[w \mid \alpha]).$$

(The first of these inequalities follows from noting that in M, the players play D with probability at least ϵ'', the second by noting that $\lambda \cdot w \leq \lambda \cdot (1,1)$ by construction.) Hence we will be finished if we can show that there is a constant K such that for all $\alpha \notin M$,

$$\lambda \cdot (1,1) > K \geq \lambda \cdot ((1-\delta)g(\alpha) + \delta E[w \mid \alpha]).$$

(Of course, K is determined by the welfare loss that results from punishment when both players are playing C with probability at least ϵ''.)

Player 1's incentive constraint is:

$$d \leq \{\pi(x' \mid D, \alpha_2) - \pi(x' \mid C, \alpha_2)\}(w_1(x'') - w_1(x')),$$

which implies that

$$\begin{aligned} v_1 &= (1-\delta)g_1(C,\alpha_2) + \delta E[w_1 \,|\, C, \alpha_2] \\ &= (1-\delta)g_1(C,\alpha_2) + \delta\left(w_1(x'') - \pi(x' \,|\, C,\alpha_2)(w_1(x'') - w_1(x'))\right) \\ &\leq (1-\delta)g_1(C,\alpha_2) + \delta\left(w_1(x'') - \frac{d}{L^{\alpha_2}-1}\right) \end{aligned}$$

where $L^{\alpha_2} = \frac{\pi(x' \,|\, D,\alpha_2)}{\pi(x' \,|\, C,\alpha_2)} > 1$ by definition of M. Similarly we get that

$$v_2 \leq (1-\delta)g_2(\alpha_1, C) + \delta\left(w_2(x'') - \frac{d}{L^{\alpha_1}-1}\right).$$

From this it follows that

$$\begin{aligned} \lambda \cdot v \leq \lambda_1 &\left((1-\delta)g_1(C,\alpha_2) + \delta\lambda_1 w_1(x'') - \delta\frac{d}{L^{\alpha_2}-1}\right) \\ &\quad + \lambda_2\left((1-\delta)g_2(\alpha_1,C) + \delta\lambda_2 w_2(x'') - \delta\frac{d}{L^{\alpha_1}-1}\right) \\ &\leq \lambda\cdot(1,1) - \delta(\lambda_1+\lambda_2)\frac{d}{L-1} \end{aligned}$$

where $L = \frac{\pi_{x'}(D,C)}{\pi_{x'}(C,C)}$. The inequality follows since $\lambda_1, \lambda_2 > 0$, $g_1(\cdot), g_2(\cdot) \leq 1$, $\lambda \cdot w(x'') \leq \lambda \cdot (1,1)$, and $1 < L^{\alpha} \leq L$ for all $\alpha \in M$. This establishes the result (with $K = \lambda\cdot(1,1) - \delta(\lambda_1+\lambda_2)\frac{d}{L-1}$.

Note the idea here is simple, but dealing with mixed strategies in a neat way requires a little work.

(c) The surprising result of this part is that as $n \to \infty$, so that the 'informativeness' of the signals are apparently decreasing, the size of the set of PPE payoffs actually increases and in fact converges to the whole feasible set:

$$\lim_{n\to\infty}\lim_{\delta\to 1} E(\delta) = \{v \,|\, v_1, v_2 \geq 0\} \cap \mathrm{Hull}((0,0), (1+d,-h), (1,1), (-h, 1+d)).$$

Denote this last set by $\bar{V}$.

To prove this, we find certain points that are contained in the set $H(\lambda)$ for different λ and show that the convex hull of these points converges to $\bar{V}$ as $n \to \infty$.

Case 1: $\lambda_1, \lambda_2 \leq 0$ (and $\lambda \neq (0,0)$): then $(0,0) \in H(\lambda)$.

Easy, $(0,0)$ is the static Nash equilibrium so it is sustained by $w_i \equiv 0$.

Case 2: $\lambda_1 > 0 \geq \lambda_2$: then $g(D,C) = (1+d,-h) \in H(\lambda)$.

The intuition for why (D,C) is supportable is that in this case, either player deviating makes x'' more likely, so we can just punish both players when x'' occurs (and this doesn't lead to inefficiency since $\lambda_1, \lambda_2 \leq 0$

implies this can be done through 'side payments'). More formally, the incentive constraints are:

$$(1-\delta)g_1(D,C) + \delta E[w_1 \mid D,C] \geq (1-\delta)g_1(C,C) + \delta E[w_1 \mid C,C]$$
$$(1-\delta)g_2(D,C) + \delta E[w_2 \mid D,C] \geq (1-\delta)g_2(D,D) + \delta E[w_2 \mid D,D]$$

or

$$\frac{1-\delta}{\delta}d \geq \gamma(w_1(x'') - w_1(x'))$$
$$\frac{1-\delta}{\delta}h \geq \beta(w_2(x') - w_2(x''))$$

We can satisfy $\lambda \cdot w = \lambda \cdot g(D,C)$ by taking $\tilde{w}_1(x'') = \tilde{w}_2(x'') = 0$, $\tilde{w}_1(x') = -\frac{\lambda_2}{\lambda_1}\tilde{w}_2(x')$. Then define $w(x) = \tilde{w}(x) + (1+h, -d)$ for $x \in \{x', x''\}$. Then we just need to take $\tilde{w}_2(x')$ large enough to satisfy the IC constraints.

Case 3: $\lambda_2 > 0 \geq \lambda_1$: then $g(C,D) = (-h, 1+d) \in H(\lambda)$.

This is analogous to Case 2.

Case 4: $\lambda_1 \geq \lambda_2 > 0$. Define $\epsilon'_n = \left(1 + \frac{1-\delta}{\delta}\frac{h}{d}\right)\frac{\gamma_n}{\beta_n + \gamma_n}$. Then if $\frac{\gamma_n}{\beta_n}$ is small enough, then we claim that $g(D_{\epsilon'_n}, C) \in H(\lambda)$ and $g(D,C) \in H(\lambda)$.

It's sufficient to show that if $\epsilon \in [\epsilon'_n, 1]$ then (D_ϵ, C) is enforceable. The key intuition is that if $\epsilon > \frac{\gamma}{\beta+\gamma}$ then the deviation that is profitable for either player today (it is D for each) affects the signal asymmetrically. Deviation to D by player 1 makes x' *more* likely, deviation to D by player 2 makes x' *less* likely. Hence we can just transfer payoff from player 2 to player 1 when x'' occurs and this will not result in inefficiency. However, it's not quite this simple since if $\epsilon < 1$ then player 1 must be indifferent between C and D, which determines the size of this transfer. Then we have to check that 2's IC constraint is satisfied. For this to be the case, two things have to happen: (a) 1's mixing probability can't be too close to $\frac{\gamma}{\beta+\gamma}$ which is where the signal becomes completely uninformative to 2's action, and (b) the transfer must be large enough, which happens when $\gamma \to 0$, so that the signal becomes very insensitive to 1's defection and so a large transfer is needed to keep 1 indifferent, which leads to 2's IC constraint being satisfied.

Formally, the IC constraints are that 1 is indifferent and 2 doesn't want to play D:

$$\frac{1-\delta}{\delta}d = \gamma(w_1(x'') - w_1(x')) \text{ and}$$
$$\begin{aligned}\frac{1-\delta}{\delta}h &\leq [\pi(x'' \mid D_\epsilon, D) - \pi(x'' \mid D_\epsilon, C)]\,(w_2(x') - w_2(x'')) \\ &= [-(1-\epsilon)\gamma + \epsilon\beta](w_2(x') - w_2(x'')).\end{aligned}$$

Choose $\tilde{w}_1(x') = \tilde{w}_2(x') = 0$, $\tilde{w}_1(x'') = \frac{1-\delta}{\delta}\frac{d}{\gamma}$, and $\tilde{w}_2(x'') = -\frac{\lambda_1}{\lambda_2}\frac{1-\delta}{\delta}\frac{d}{\gamma}$. Then define $w(x) = \tilde{w}(x) + g(D_\epsilon, C)$ for $x \in \{x', x''\}$. This ensures 1's indifference and that $\lambda \cdot w = \lambda \cdot g(D_\epsilon, C)$. Hence we just need to check whether 2's IC constraint holds. Substituting into the RHS of the constraint, we get:

$$\begin{aligned}
[-(1-\epsilon)\gamma + \epsilon\beta](w_2(x') - w_2(x'')) &= [\epsilon(\gamma+\beta) - \gamma]\frac{\lambda_1}{\lambda_2}\frac{d}{\gamma} \\
&\geq d\left[\epsilon\frac{\gamma+\beta}{\gamma} - 1\right] \text{ since } \lambda_1 \geq \lambda_2 > 0 \\
&\geq \frac{1-\delta}{\delta}h \quad \text{if } \epsilon \geq \epsilon' = \left(1 + \frac{1-\delta}{\delta}\frac{h}{d}\right)\frac{\gamma}{\beta+\gamma}.
\end{aligned}$$

It follows that for n large, using the result of Fudenberg and Levine that $\lim_{\delta\to 1} E(\delta) = \cap_\lambda H(\lambda)$, we have that

$$\lim_{\delta\to 1} E(\delta) \supseteq \{v \mid v_1, v_2 \geq 0\} \cap \text{Hull}((0,0), (1+d, -h), (-h, 1+d), g(D_{\epsilon'_n}, C), g(C, D_{\epsilon'_n})).$$

Since $g(D_{\epsilon'_n}, C) \to (1,1)$ and $g(C, D_{\epsilon'_n}) \to (1,1)$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\lim_{\delta\to 1} E(\delta) = \bar{V}$$

as claimed.

(d) What's going on here? The point of the question is that one needs to be very careful about the idea of 'less informative'. Here one can show that as $\gamma_n, \beta_n \to 0$, the signal structure is not becoming less informative in the sense of Blackwell. Note that the failure of pairwise identifiability means that the conditions for the folk theorem of FLM (1994) are not satisfied; it is therefore interesting that at least in the limit (in part (c)), we do get the folk theorem back (and the failure for finite n can be arbitrarily small).

Question 4.

(a) No.

Denote the type space of player i by Z_i, her true type by $z_i \in Z_i$, her move $y_i \in Z_i$. A pure action of player i is a map $\alpha_i : Z_i \to Z_i$ (so that $\alpha_i(z_i) = y_i$) since by assumption the action space coincides with the type space.

The definition of a game with a product structure (see FLM(1994) page 1027, definition 7.1) requires that for all i and α, the marginal distribution of y_i satisfies

$$\pi_i(y_i|\alpha) = \pi_i(y_i|\alpha_i, \alpha_{-i}) = \pi_i(y_i|\alpha_i, \alpha'_{-i})$$

for all y_i and α'_{-i}; that is, that the marginal only depends on player i's strategy α_i. Evidently, this holds in the game of the problem, since

$$\pi_i(y_i|\alpha) = p(\alpha_i^{-1}(y_i))$$

independently of α_{-i}. (Note the slight abuse of notation, as $\alpha_i(.)$ in fact maps from Z_i, not from the overall type space $\Pi_i\ Z_i$ where the measure $p(.)$ is defined).

Product structure also requires for each α that

$$\pi(y|\alpha) = \prod_i \pi_i(y_i|\alpha_i)$$

for all y. In our game, this condition is equivalent to

$$p(\alpha_1^{-1}(y_1) \cap ... \cap \alpha_n^{-1}(y_n)) = \prod_i p(\alpha_i^{-1}(y_i)).$$

This condition holds for all α and y if and only if $p(.)$ is a product measure. In our case it is not, hence the game does not have a product structure.

(b) No.

Let α be the strategy of truthful reporting, then $\alpha_i(z_i) = z_i$. Because each player has two possible types, each player has four possible actions (contingent plans) and the matrix $\Pi_i(\alpha_{-i})$ has 4 rows. Because there are only two players, and each has two possible moves, there are four possible outcomes, hence $\Pi_i(\alpha_{-i})$ has four columns.

Denote the possible types of player i by high (H) and low (L), that is, let $Z_i = \{H, L\}$. Similarly, let $Z_j = \{\overline{H}, \overline{L}\}$. Then the rows of $\Pi_i(\alpha_{-i})$ correspond to the four possible contingent plans of player i, namely HH, HL, LH and LL, where truthful reporting is the plan HL.

The columns correspond to the four possible outcome profiles, namely $H\overline{H}$, $H\overline{L}$, $L\overline{H}$ and $L\overline{L}$, where the first letter denotes the move of player i.

The distribution $p(.)$ assigns probabilities to the type profiles $H\overline{H}$, $H\overline{L}$, $L\overline{H}$ and $L\overline{L}$. Denote $p(H\overline{H}) = p_1$, $p(H\overline{L}) = p_2$, $p(L\overline{H}) = p_3$ and $p(L\overline{L}) = p_4$. Then the matrix $\Pi_i(\alpha_{-i})$ has the following form

$$\begin{array}{c c} & \begin{array}{cccc} H\overline{H} & H\overline{L} & L\overline{H} & L\overline{L} \end{array} \\ \begin{array}{c} HH \\ HL \\ LH \\ LL \end{array} & \left[\begin{array}{cccc} p_1 + p_3 & p_2 + p_4 & 0 & 0 \\ p_1 & p_2 & p_3 & p_4 \\ p_3 & p_4 & p_1 & p_2 \\ 0 & 0 & p_1 + p_3 & p_2 + p_4 \end{array} \right] \end{array}.$$

To understand why, consider the first row. If player i plays HH, that is, she plays H no matter what her type is, then the outcome can only be $H\overline{H}$ or $H\overline{L}$. Since player j is truthfully reporting, the outcome will be $H\overline{H}$ if and only if player 2's type is $\overline{H}$, which happens with probability $p_1 + p_3$. Similarly, $H\overline{L}$ happens with probability $p_2 + p_4$.

In the second row, both players are playing truthful reporting, hence the distribution on outcomes is the same as the type distribution. One can similarly check the other two rows.

Now inspection reveals that $\Pi_i(\alpha_{-i})$ has rank 3 for generic distributions. Indeed, the sum of the first and the last rows is the same as the sum of the second and the third rows, hence the rank cannot be four; but it is easy to verify that there is no other linear dependence among the rows unless we make special assumptions about the distribution (e.g., that it is a product measure).

Therefore the rows of $\Pi_i(\alpha_{-i})$ span a three dimensional subspace in R^4.

Now a completely symmetric argument proves that the rows of $\Pi_j(\alpha_{-j})$ also span a three dimensional subspace. The intersection of these two subspaces in R^4 is a subspace that is at least two dimensional. Thus there are at least two linear dependences between the rows of $\Pi_i(\alpha_{-i})$ and $\Pi_j(\alpha_{-j})$. It follows that $\Pi_{ij}(\alpha)$ is not pairwise identifiable.

(c) Suppose player 3 always plays the same (Up), and players 1 and 2 play truthful reporting. This profile (α) will not be pairwise identifiable for players 1 and 2 for generic distributions.

The matrix $\Pi_1(\alpha_{-1})$ will continue to have four rows representing the actions of player 1. It will have eight columns corresponding to the eight possible outcomes. However, player 3 never goes Down, so four of these eight columns (corresponding to the outcomes where player 3 moves Down) contain only zeros. The other four columns in turn are identical to the four columns of $\Pi_i(\alpha_{-i})$ in part (b) above. Indeed, it is as if we were in a two-player game where only the first two players are allowed to move, and player 3 is constrained to play Up no matter what.

Hence the same arguments used in part (b) apply here too, and the profile is not pairwise identifiable.

(d) Here let $p_1 = p(HHH)$, $p_2 = p(HHL)$, $p_3 = p(HLH)$, $p_4 = p(HLL)$, $p_5 = p(LHH)$, $p_6 = p(LHL)$, $p_7 = p(LLH)$ and $p_8 = p(LLL)$. Then the matrix $\Pi_1(\alpha_{-1})$ takes the form

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ p_5 & p_6 & p_7 & p_8 & p_1 & p_2 & p_3 & p_4 \\ p_1+p_5 & p_2+p_6 & p_3+p_7 & p_4+p_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1+p_5 & p_2+p_6 & p_3+p_7 & p_4+p_8 \end{pmatrix}$$

This clearly has rank 3: as before the sum of the first and second rows equals the sum of the third and fourth rows. As in part (b) it's easy to check there are no other linear dependences for generic p. Similarly $\Pi_2(\alpha_{-2})$ has rank 3. Its form is

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ p_3 & p_4 & p_1 & p_2 & p_7 & p_8 & p_5 & p_6 \\ p_1+p_3 & p_2+p_4 & p_1+p_3 & p_2+p_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_5+p_7 & p_6+p_8 & p_5+p_7 & p_6+p_8 \end{pmatrix}$$

Stacking these matrices to get Π_{12}, we notice that the first and fifth rows are the same and elementary matrix operations allow us to eliminate the second and sixth rows: after moving the zero rows to the bottom, we see Π_{12} is row equivalent to:

$$\begin{pmatrix} p_1 & p_2 & p_3 & p_4 & p_5 & p_6 & p_7 & p_8 \\ p_1+p_3 & p_2+p_4 & p_1+p_3 & p_2+p_4 & 0 & 0 & 0 & 0 \\ p_1+p_5 & p_2+p_6 & p_3+p_7 & p_4+p_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p_1+p_5 & p_2+p_6 & p_3+p_7 & p_4+p_8 \\ 0 & 0 & 0 & 0 & p_5+p_7 & p_6+p_8 & p_5+p_7 & p_6+p_8 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

It's clear this is of rank 5 for generic p. Since $5 = 3 + 3 - 1$, we have established pairwise identifiability.