**14.126**
**Fall 2003**
**Problem Set 3**
**Suggested Solutions**

**Question 1.**

(a) I'll assume that $w > v > 0$ here, not just $w > v$ as written in the question: otherwise this is not a Hawk-Dove game.

It's easy to see this game has three Nash equilibria for the given parameters: two asymmetric pure strategy equilibria $(H, D)$ and $(D, H)$ together with a symmetric mixed strategy equilibrium. Suppose player 2 puts probability $x$ on $H$; then player 1's expected payoff to $H$ is $x\frac{v-w}{2} + (1-x)v$, and to $D$, $(1-x)\left(\frac{v}{2} - t\right)$. Equating these shows that $x = \frac{v+2t}{w+2t}$ which is strictly between 0 and 1 as $w > v > 0$ and $t > 0$. Thus the only symmetric equilibrium is for both players to play $H$ with probability $x_H = \frac{v+2t}{w+2t}$.

Only the symmetric equilibrium is a candidate to be an ESS, since ESS are Nash equilibrium strategies. To check that this is the case for our mixed NE, it suffices to observe that for any strategy $y$ with $y_H \neq x_H$, we have $u(x-y, x) = 0$ since $x$ is such that the opponent is indifferent between her pure strategies when facing $x$. Also

$$\begin{aligned}
u(x-y, y) &= \begin{pmatrix} x_H - y_H & (1-x_H) - (1-y_H) \end{pmatrix} \begin{pmatrix} \frac{v-w}{2} & v \\ 0 & \frac{v}{2} - t \end{pmatrix} \begin{pmatrix} y_H \\ 1 - y_H \end{pmatrix} \\
&= (x_H - y_H)\left(\frac{v-w}{2} y_H + \frac{v+2t}{2}(1-y_H)\right) \\
&= \frac{1}{2}(x_H - y_H)(v + 2t - (w+2t)y_H) \\
&= \frac{1}{2}(x_H - y_H)^2(w+2t) \\
&> 0.
\end{aligned}$$

Hence the unique symmetric NE is the only ESS.

(b) First note that if $w > v > 0$ and $t > 0$ then $(B, B)$ is a strict Nash equilibrium and hence evolutionarily stable. Is this all? More formally, the condition for $B$ to be an ESS is just that

$$\begin{aligned}
& u(B,B) > u(pH + qD + (1-p-q)B, B) \\
\text{or} \quad & u(B,B) = u(pH + qD + (1-p-q)B, B) \\
\text{and } & u(B, pH + qD + (1-p-q)B) \\
& \quad > u(pH + qD + (1-p-q)B, pH + qD + (1-p-q)B)
\end{aligned}$$

for all $p, q \in [0, 1]$ with $p + q > 0$.

Now

$$\min_{p,q} u(B,B) - u(pH + qD + (1-p-q)B, B)$$
$$= \min_{p,q} pu(B-H,B) + qu(B-D,B)$$
$$= \min_{p,q} p\frac{w-v}{4} + q\frac{v+2t}{4}$$

which is nonnegative iff $w \geq v$ and $v + 2t \geq 0$. Equality holds for $p$ and $q$ not both zero only if $w = v$ or $v + 2t = 0$.

If $w = v$ or $v + 2t = 0$ then we need to calculate

$$\begin{aligned}
&u(B, pH + qD + (1-p-q)B) - u(pH + qD + (1-p-q)B, pH + qD + (1-p-q)B) \\
&= pu(B-H, pH + qD + (1-p-q)B) + qu(B-D, pH + qD + (1-p-q)B) \\
&= p\left(-p\frac{v-w}{4} - q\frac{v+2t}{4} - (1-p-q)\frac{v-w}{4}\right) \\
&\qquad + q\left(p\frac{v-w}{4} + q\frac{v+2t}{4} + (1-p-q)\frac{v+2t}{4}\right) \\
&= p(1-2q)\frac{w-v}{4} + q(1-2p)\frac{v+2t}{4}.
\end{aligned}$$

If $w = v$ and $v + 2t > 0$ we need the right side to be strictly positive for $p > 0$ and $q = 0$, which is not the case. If $w > v$ and $v + 2t = 0$ we need the right side to be strictly positive for $p = 0$ and $q > 0$, which is again not the case. Finally, if $w = v$ and $v + 2t = 0$, irrespective of $p$ and $q$ the right side is zero. Thus in no case with equality is $(B, B)$ an ESS.

In summary, $(B, B)$ is an ESS iff $w > v$ and $v + 2t > 0$.

**Question 2.**

(a) Denote a (mixed) strategy for player 1 by $p = \{p_1, p_2, \ldots, p_{100}\}$, where $p_i$ is the probability that player 1 offers $a = i/100$. Let $\underline{i}$ be the lowest value of $i$ such that $p_i > 0$.

It's clear that Player 2's best response to $p$ consists of those strategies that put positive probability only on those $b$ that are weakly less than $\underline{a} = \underline{i}/100$ (otherwise Player 2 is rejecting positive payoffs unnecessarily). That implies that if $p_i > 0$, then Player 2 accepts $i/100$ in equilibrium with probability 1, so the payoff to playing $a = i/100$ is $1 - a$. But Player 1 must be indifferent between all strategies she uses with positive probability; this implies that Player 1's strategy must be a pure strategy.

Let $a$ be Player 1's strategy; then by the previous argument we know that Player 2 plays strategies $b \leq a$ with probability 1 and equilibrium payoffs are $(1-a, a)$. Denote 2's strategy by $q = \{q_1, q_2, \ldots, q_{100}\}$; then $q_i = 0$ for $i > 100a$. It's clear that 2 is playing a best response to 1's strategy. Therefore we just need to check the condition for 1 to prefer to play $a$ rather than deviate. Deviating to $a' > a$ gives 1 a payoff of $1 - a' < 1 - a$. Deviating to $a' < a$ gives 1 a payoff of

$$(1 - a')\Pr(b \le a') = (1 - a')\sum_{i=1}^{100a'} q_i$$

so for this to be an equilibrium, it must be that

$$\sum_{i=1}^{100a'} q_i \le \frac{1 - a}{1 - a'} \text{ for all } a' < a. \tag{1}$$

Thus the set of all Nash equilibria is all strategy pairs $(a, q) \in \{0.01, \ldots, 1\} \times \Delta$ (where $\Delta$ is the unit simplex in $\mathbb{R}^{100}$) that satisfy (1). (Note the set of all pure strategy Nash equilibria is clearly just the set of all pairs $(a, a)$ where $a \in \{0.01, \ldots, 1\}$).

(b) To apply the replicator dynamics, we are only interested in pure strategies. Since ASS are NE strategies, the only candidate ASS are the pairs $(a, b)$ where $a = b$.

First, note that $a = b = 0.01$ is asymptotically stable. This follows from observing that it is a strict Nash equilibrium, which we now check. Playing pure strategy $(a', b')$ gives a player who is assigned to be player 1 a payoff of $1 - a' \le 0.99$ since $a'$ will be accepted for sure against a player playing $(0.01, 0.01)$; if assigned to be player 2, the payoff is 0.01 if $b' = 0.01$ and 0 otherwise. Thus expected payoff of playing $(a', b')$ is $\frac{1}{2}(1-a'+0.01\cdot\mathbf{1}(b' = 0.01))$, which is clearly strictly less than 0.5 if $a' \ne 0.01$ or $b' \ne 0.01$.

Next, observe that if $a > 0.01$ then $(a, a)$ is not asymptotically stable. Consider a small population (size $\epsilon$) of mutants who play $(a, a - 0.01)$. Now we need to observe that whenever two such agents meet, the outcome of the game is that $a$ is offered and accepted, so expected payoff to any player is $\frac{1}{2}(a + (1 - a)) = \frac{1}{2}$. That is, the population shares of the mutants and the non-mutants remain constant under replicator dynamics. It follows that $(a, a)$ is not asymptotically stable since any neighborhood of $(a, a)$ contains a strategy that does not vanish asymptotically.

**Problem 3**

(a) We will show that the unique rationalizable strategy profile is the Nash equilibrium profile, in which player $i$ sets

$$q_i^* = \begin{cases} \frac{a+2b_i-b_{-i}}{3} & \text{if } \min\{a + 2b_1 - b_2, a + 2b_2 - b_1\} \ge 0; \\ 0 & \text{if } a + 2b_i - b_{-i} < 0; \\ \frac{a+b_i}{2} & \text{if } a + 2b_i - b_{-i} \ge 0 > a + 2b_{-i} - b_i. \end{cases}$$

To find the rationalizable strategies, as usual we proceed through iterated elimination of strictly dominated strategies (note that strategies that are (iteratively) dominated by any pure strategy cannot be rationalizable, although the equivalence of rationalizability to surviving iterated elimination of strictly dominated strategies is true in general only in games with finitely many pure strategies). Notice that as usual in a Cournot game, the players' quantity choices are strategic substitutes (if $q_{-i}$ increases then player $i$'s best response, $BR(q_{-i})$ decreases, since the payoff to playing $q_i$ is $q_i(a + b_i - q_i - q_{-i})$, which leads to $BR(q_{-i}) = \max\left\{0, \frac{1}{2}(a + b_i - q_{-i})\right\}$).

- We assume that it isn't feasible for players to choose $q.$ negative.[1]
- Given that player $-i$ is choosing a strategy $q_{-i} \in [0, \infty)$, and given the strategic substitute property, it's strictly dominated for player $i$ to choose $q_i \notin [BR(\infty), BR(0)] = [0, \max\{0, \frac{a+b_i}{2}\}] \equiv I_i^1$. Similiarly for $-i$.
- Given that player $-i$ is choosing a strategy $q_{-i} \in [0, \max\{0, \frac{a+b_i}{2}\}]$, and given the strategic substitute property, it's strictly dominated for player $i$ to choose $q_i \notin [BR(\max\{0, \frac{a+b_i}{2}\}), BR(0)] = [\max(0, \frac{b_i - b_{-i}}{2}), \max\{0, \frac{a+b_i}{2}\}] \equiv I_i^2$. Similiarly for $-i$.
- An easily formalizable induction now shows that there exist decreasing sequences of closed intervals $I_i^1 \supseteq I_i^2 \supseteq \ldots$ for $i = 1, 2$ such that for each $n$, $I_i^n = BR(I_{-i}^{n-1})$, and such that iterated elimination of strictly dominated strategies shows that any rationalizable $q_i \in I_i$. To complete the proof that there is at most one rationalizable strategy pair $(q_1, q_2)$ requires observing that the inductive construction does not 'stop' before each $I.$ decreases to a singleton; there are multiple cases to check here and it's easiest to argue graphically (as in recitation).
- The Nash equilibrium of the game (which is rationalizable) can be found by solving for the intersection of the two best response correspondences; it's easy to check this leads to $q_i = q_i^*$ as claimed.

Hence there is a unique rationalizable strategy profile in this game.

If $\min\{a + 2b_1 - b_2, a + 2b_2 - b_1\} \geq 0$ then aggregate production is $Q = q_1 + q_2 = \frac{2a+b_1+b_2}{3}$, so the true price is $P = \frac{a-b_1-b_2}{3}$. It follows that actual payoffs are

$$u_1(b_1, b_2) = \frac{1}{9}(a + 2b_1 - b_2)(a - b_1 - b_2) \tag{2}$$

$$u_2(b_1, b_2) = \frac{1}{9}(a + 2b_2 - b_1)(a - b_1 - b_2). \tag{3}$$

If $\max\{a + 2b_1 - b_2, a + 2b_2 - b_1\} < 0$ then $q_1 = q_2 = 0$ and payoffs are $u_1 = u_2 = 0$.

If $a + 2b_1 - b_2 \geq 0 > a + 2b_2 - b_1$ then $q_1 = \frac{a+b_1}{2}$ and $q_2 = 0$, so $Q = q_1$, and $P = \frac{a-b_1}{2}$. Payoffs are $u_1 = \frac{1}{4}(a + b_1)(a - b_1)$ and $u_2 = 0$. Conversely if $a + 2b_2 - b_1 \geq 0 > a + 2b_1 - b_2$ then $u_1 = 0$ and $u_2 = \frac{1}{4}(a + b_2)(a - b_2)$.

(b) To find the NE of $\Gamma$, first observe that either player can guarantee herself at least a payoff of 0 by choosing $b_i$ large and negative enough that $a + 2b_i - b_{-i} <$

[1] Note if we do not make some such assumption then it's not possible to begin the iterated elimination of strictly dominated strategies: $BR((-\infty, \infty)) = (-\infty, \infty)$.

$0 \leq a + 2b_{-i} - b_i$. Note that if $b_1 > a$ then player 1 is getting a negative payoff (either $u_1 = \frac{1}{4}(a+b_1)(a-b_1) < 0$ if $a+2b_2-b_1 < 0$ or, if $a+2b_2-b_1 \geq 0$, then $u_1 = \frac{1}{9}(a+2b_1-b_2)(a-b_1-b_2) < 0$ since if $a+2b_2-b_1 \geq 0$ then $b_2 \geq \frac{1}{2}(b_1-a)$, so $a-b_1-b_2 \leq \frac{3}{2}(a-b_1) < 0$). Hence no NE has $b_1 > a$. Similarly no NE has $b_2 > a$.

Next, if $b_2 < a$ then player 1 can get a positive payoff by choosing $b_1 = 0$ (if $a + 2b_2 \geq 0$ this gives $u_1 = \frac{1}{9}(a-b_2)^2$ while if $a+2b_2 < 0$ it gives $u_1 = \frac{1}{4}a^2$. Similarly if $b_1 < a$ then player 2 can ensure $u_2 > 0$). Also, if $b_2 = a$ then player 1 is getting a negative payoff unless $b_1 \leq 0$, but in that case, player 2 is getting $u_2 = 0$ and can deviate to ensure $u_2 > 0$ as above. Hence there are no NE with $b_1 = a$ or $b_2 = a$. But in this case any NE must have $b_1 < a$ and $b_2 < a$; it follows that both players must be getting positive payoffs in equilibrium since both can find a deviation that gives them positive payoff. But the only $(b_1, b_2)$ for which $u_1, u_2 > 0$ satisfy $\min\{a+2b_1-b_2, a+2b_2-b_1\} \geq 0$ and have payoffs given by (2) and (3). But in this range, we can use FOC to find that any equilibrium must satisfy

$$\begin{aligned} 2(a-b_1-b_2) - (a+2b_1-b_2) = 0 \quad &\Longrightarrow \quad a - 4b_1 - b_2 = 0 \\ 2(a-b_2-b_1) - (a+2b_2-b_1) = 0 \quad &\Longrightarrow \quad a - b_1 - 4b_2 = 0 \end{aligned}$$

which has unique solution $b_1 = b_2 = b^* = a/5$. Hence this is the unique NE of $\Gamma$.

To show that the replicator dynamics in $\Gamma$ converges to $b^*$, we would like to apply the result of Samuelson and Zhang (that the share of any strategy $s$ that is eliminated by iterated pure strategy strict dominance converges asymptotically to zero).

- First, observe that playing $b_i > a$ is strictly dominated by playing $b_i' = -a$ which gives a payoff of 0 for sure. Thus we can restrict attention to strategies $b_i \in (-\infty, a]$.
- Next, for $b_{-i} \in (-\infty, a]$, $b_i = a$ is dominated by $b_i' = 0$ ($b_i = a$ gives $u_i = 0$ if $b_{-i} \leq 0$ and $u_i < 0$ if $0 < b_{-i} \leq a$, while $b_i = 0$ gives $u_i > 0$ for $b_{-i} < a$ and $u_i = 0$ for $b_{-i} = a$). Thus we can restrict attention to strategies $b_i \in (-\infty, a)$.
- Next, for $b_{-i} \in (-\infty, a)$, $b_i \leq -a$ is dominated by $b_i' = 0$ ($b_i \leq -a$ gives a payoff of at most 0, while $b_i = 0$ gives a strictly positive payoff). Thus we can restrict attention to strategies $b_i \in [0, a) \subset [0, a] \equiv I_1$.
- Given that both players choose strategies in $I_1$, payoffs are given by (2) and (3). This game has a strong formal similarity to the game of part (a), and a very similar argument to that given there shows that iterated elimination of strictly dominated strategies eliminates all strategies except the Nash equilibrium $b^*$.

It follows that the replicator dynamics converge to $b^*$.

(c) There is a slight ambiguity in the question about what the payoff functions that are used in $G(b_1, b_2)$ are; it makes most sense to assume that the replicator dynamics in $G(b_1, b_2)$ use the perceived (biased) utility functions while those in $\Gamma$ use the true (unbiased) utility functions.[2]

In this case the result of Samuelson and Zhang applies to the replicator dynamics in $G(b_1, b_2)$ for any $(b_1, b_2)$ and shows that the replicator dynamics converge to the unique strategy that survives iterated elimination of strictly dominated strategies, which gives strategies and payoffs as calculated in part (a). Thus replicator dynamics under $\Gamma$ use these payoffs and the argument in part (b) applies to show that there is convergence to $b^*$.

If the replicator dynamics in $G(b_1, b_2)$ use the true payoffs rather than the perceived payoffs, then $G(b_1, b_2)$ is payoff equivalent to $G(0, 0)$ and so replicator dynamics in $G(b_1, b_2)$ converge to $q_1 = q_2 = a/3$ independent of $(b_1, b_2)$. In this case the dynamics in $\Gamma$ are constant since all strategies $b$ lead to the same payoff.

(d) There are several interesting aspects of this problem.

- Note that in the dynamics in $\Gamma$, although players optimize given their 'true' utility functions, still some over-optimism survives in equilibrium. This is really a result about commitment rather than evolution, though: it's true already in the Nash equilibrium of $\Gamma$. Committing to be over-optimistic ($b_i > 0$) means I can commit to produce more in the Cournot game, which is a good thing to do at the margin because of the strategic substitutes property. Note this interpretation doesn't survive if the replicator dynamics in $G(b_1, b_2)$ use the true payoffs, since then over-optimism is irrelevant.
- It's interesting that some of the results using finite state spaces carry over to the continuous state space setting here. The special (convex/concave) structure of the Cournot game has a lot to do with this.

**Problem 4**

Suppose fraction $x_t$ of the population are playing $A$ at time $t$. Note that $x_t \in \mathcal{S} \equiv \{0, \frac{1}{N}, \ldots, 1\}$.

To calculate the ergodic distribution, we first need the transition probabilities $\Pr\left(x_{t+1} = x_t + \frac{1}{N} \,|\, x_t\right)$ and $\Pr\left(x_{t+1} = x_t - \frac{1}{N} \,|\, x_t\right)$ (we'll abbreviate these to $\Pr\left(x_t \pm \frac{1}{N} \,|\, x_t\right)$). An agent playing strategy $A$ is selected with probability $x_t$ and switches strategies when selected if

- she switches without observation: probability $\epsilon$;

[2] There is also an ambiguity about the starting point for the replicator dynamics in each tier of the game: it's neatest to assume we start with a population with full support rather than with the limit of the previous time we played the game since the replicator dynamics cannot increase the share of a strategy from 0 to a positive value.

- she observes an agent playing $A$ and draws $U > u_1(A, A) = 1$: probability zero;
- she observes an agent playing $B$ and draws $U > u_1(A, B) = 0$: probability $(1 - x_t)(1 - \epsilon)$.

Hence the probability that a randomly selected agent switches from $A$ to $B$ is

$$\Pr\left(x_t - \frac{1}{N} \,|\, x_t\right) = x_t[\epsilon + 0 + (1 - x_t)(1 - \epsilon)] \\ = x_t(1 - x_t + \epsilon x_t). \tag{4}$$

Similarly the probability that a randomly selected agent switches from $B$ to $A$ is

$$\Pr\left(x_t + \frac{1}{N} \,|\, x_t\right) = (1 - x_t)\left[\epsilon + x_t(1 - l)(1 - \epsilon) + \frac{1}{2}(1 - x_t)(1 - \epsilon)\right] \\ = (1 - x_t)\left[\frac{1}{2} + x_t\left(\frac{1}{2} - l\right) + \epsilon\left(\frac{1}{2} - x_t\left(\frac{1}{2} - l\right)\right)\right]. \tag{5}$$

It's clear that either $\Pr(x_{t+1} \,|\, x_t) = 0$ for $x_{t+1} \notin \{x_t \pm \frac{1}{N}\}$ since this is a birth-death process.

(a) As $\epsilon \to 0$, $\Pr\left(\frac{N-1}{N} \,|\, 1\right) = \epsilon \to 0$; however, for all other $x_t \in \mathcal{S}$ with $\frac{2}{N} \leq x_t \leq \frac{N-1}{N}$, then $\Pr\left(x_t - \frac{1}{N} \,|\, x_t\right) \to x_t(1 - x_t) \neq 0$ as $\epsilon \to 0$. Also, for any $x_t \in \mathcal{S}$ with $x_t \leq \frac{N-1}{N}$, we have that $\Pr\left(x_t + \frac{1}{N} \,|\, x_t\right) \to (1 - x_t)\left[\frac{1}{2} + x_t\left(\frac{1}{2} - l\right)\right] \neq 0$. It follows immediately that the limit as $\epsilon \to 0$ of the ergodic distribution for $x_t$ is the point mass on $x_t = 1$ (that is, where all players are playing $A$.

(b) As $N \to \infty$ with $\epsilon$ fixed (and sufficiently small), the ergodic distribution converges (weakly) to a point mass at the point $x_t = x$ that satisfies

$$\Pr\left(x_t - \frac{1}{N} \,|\, x_t\right) = \Pr\left(x_t + \frac{1}{N} \,|\, x_t\right),$$

that is,

$$x(1 - x + \epsilon x) = (1 - x)\left[\frac{1}{2} + x\left(\frac{1}{2} - l\right) + \epsilon\left(\frac{1}{2} - x\left(\frac{1}{2} - l\right)\right)\right].$$

which can be simplified to give

$$\frac{1}{2}(1 + \epsilon) + x[-1 - l - \epsilon + \epsilon l] + x^2\left(\frac{1}{2} + l\right)(1 - \epsilon) = 0. \tag{6}$$

or[3]

$$x = \frac{1 + l + \epsilon - \epsilon l - \left((1 + l + \epsilon - \epsilon l)^2 - 4(1 + \epsilon)(1 - \epsilon)\left(\frac{1}{2} - l\right)\right)^{1/2}}{(1 + 2l)(1 - \epsilon)}.$$

[3] Take the negative sign in the quadratic formula since putting $x_t = 0$ makes the right side of (6) positive, while putting $x_t = 1$ makes it negative; hence only the smaller root lies in $[0, 1]$.

Denote this value of $x$ by $x^*$.

Note for $\epsilon \approx 0$, (6) is approximately

$$\frac{1}{2} - x(1+l) + x^2\left(\frac{1}{2}+l\right) = 0$$

which has roots $x = 1$ and $x = \frac{1}{1+2l}$. Thus for $\epsilon \approx 0$ and $N \to \infty$, the ergodic distribution is nearly a point mass at $\frac{1}{1+2l}$.

To prove this result, suppose $\pi^N$ is the ergodic distribution for some fixed $N$ (and fixed $\epsilon$), and note that since $\pi^N$ is a fixed point of the dynamical system, hence $\pi(x)\Pr(x+\frac{1}{N}\,|\,x) = \pi(x+\frac{1}{N})\Pr(x\,|\,x+\frac{1}{N})$ or

$$\frac{\pi(x+\frac{1}{N})}{\pi(x)} = \frac{\Pr(x+\frac{1}{N}\,|\,x)}{\Pr(x\,|\,x+\frac{1}{N})}. \tag{7}$$

for each $x \in \mathcal{S}\setminus\{1\}$.

Choose $\delta \in (0,\bar{\delta})$ where $\bar{\delta} > 0$ is to be defined below. Denote by $x^{**}$ the greatest integer multiple of $1/N$ not greater than $x^*$ (that is, $x^* - x^{**} \in [0, 1/N)$ and $Nx^{**}$ is an integer). Now let $k \in \mathbb{N}$ be such that $k/N$ is the least integer multiple of $1/N$ not less than $\delta$. Note this implies $k \geq N\delta$. Suppose $y \in \mathcal{S}\setminus[x^{**}-2\delta, x^{**}+2\delta]$. First suppose $y > x^{**}+2\delta$. Let

$$\rho = \frac{\Pr(x^*+\delta+\frac{1}{N}\,|\,x^*+\delta)}{\Pr(x^*+\delta,\,|\,x^*+\delta+\frac{1}{N})}$$

where the probabilities on the right are defined for any $x \in [0,1]$ via the formulae (4) and (5). Note that $0 < \rho < 1$ and that $\rho$ depends only on $l$, $\epsilon$ and $\delta$ (in particular, $\rho$ doesn't depend on $N$. Let $\bar{\delta}$ be small enough that

$$\frac{\Pr(x^*+\xi+\frac{1}{N}\,|\,x^*+\xi)}{\Pr(x^*+\xi,\,|\,x^*+\xi+\frac{1}{N})}$$

is decreasing in $\xi$ for $\xi \in [0, 2\bar{\delta}]$; it's easy to check that such a $\bar{\delta} > 0$ exists and depends only on $l$ and $\epsilon$. Now we can apply a simple induction on (7) to get that for any $x \in \mathcal{S}$ with $y - x^* \in [0,\delta)$,

$$\begin{aligned}\frac{\pi(y)}{\pi(x)} &= \frac{\pi(x^{**}+\frac{k+1}{N})}{\pi(x)}\left[\prod_{j=k+1}^{2k-1}\frac{\pi(x^{**}+\frac{j+1}{N})}{\pi(x^{**}+\frac{j}{N})}\right]\left[\prod_{j=2k}^{N(y-x^{**})-1}\frac{\pi(x^{**}+\frac{j+1}{N})}{\pi(x^{**}+\frac{j}{N})}\right]\\ &\leq 1\cdot\rho^{k-1}\cdot 1 \leq \rho^{N\delta-1}.\end{aligned}$$

That is, for $y - x^* > 2\delta$ and $x - x^* \in [0,\delta)$, we have that $\pi(y)/\pi(x) \leq \rho^{N\delta-1}$. Hence the total mass that $\pi$ puts on points in $[x^*, x^*+\delta]$ is at least

$$\rho^{N\delta-1}\frac{\delta-\frac{1}{N}}{1-x^*-2\delta+\frac{1}{N}}$$

times the total mass put on points in $[x^* + 2\delta, 1]$ (the $1/N$ terms correct for the fact that $x^* + \delta$ is not an element of $\mathcal{S}$ in general). It's clear this converges to 0 as $N \to \infty$. A similar argument shows that the ratio of the amount of mass $\pi$ puts on points in $[x^* - \delta, x^*]$ to the mass put on points in $[0, x^* - 2\delta]$ diverges to $\infty$ also. Hence for $N$ large enough, $\pi$ puts an arbitrarily small mass on points outside $[x^* - 2\delta, x^* + 2\delta]$. Since $\delta > 0$ was arbitrary, this establishes that $\pi$ converges weakly to a point mass at $x^*$ as claimed.