# 14.128 Mathematical Preliminaries

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For full definitions see SLP Chapter 3 or Luenberger, "Optimization by vector space methods".

**Vector Space** - Space in which can carry out addition and scalar multiplication

$x, y \in X \Longrightarrow x + y \in X$ and $\alpha x \in X$ for scalar $\alpha$. The operations of addition and scalar multiplication follow all the rules familiar from the real numbers.

Examples: Euclidean space $\mathbb{R}^n$, sequences on the reals $\mathbb{R}^{\mathbb{N}}$, continuous functions on an interval $[a, b]$

**Convex Sets**

A set $K$ in a vector space is convex $\Leftrightarrow \alpha x + (1 - \alpha) y \in K$ whenever $x, y \in K$ and $0 \leq \alpha \leq 1$

**Metric Space** - any space $S$ in which we can define distance:

Metric:$\rho(x, y) : S^2 \rightarrow \mathbb{R}$ satisfies:

i) $\rho(x, y) \geq 0$ ii) $\rho(x, y) = \rho(y, x)$ and iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$ TRIANGLE INEQUALITY

**Normed Vector Space** - Vector space with a metric defined by a NORM

Norm: i) $\|x\| \geq 0$ ii)$\|\alpha x\| = |\alpha| \|x\|$ and iii) $\|x + y\| \leq \|x\| + \|y\|$

defines metric $\rho(x, y) = \|x - y\|$

Examples: Euclidean norm on $\mathbb{R}^n$, $\left(\sum_{i=1}^n x_i^2\right)^{1/2}$ sup norm on the continuous functions: $\|f\| = \sup_{x \in [a,b]} |f(x)|$

**Convergence** - can define in any metric space

$\{x_n\}_{n=0}^{\infty}$ converges to $x \Leftrightarrow$ any $\varepsilon > 0$ can find $N_\varepsilon$ such that $\rho(x, x_n) < \varepsilon$ for all $n > N_\varepsilon$

**Cauchy Criterion** - convergence without defining limit

$\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence $\Leftrightarrow$any $\varepsilon > 0$ can find $N_\varepsilon$ such that $\rho(x_m, x_n) < \varepsilon$ for all $m, n > N_\varepsilon$.

Convergence$\Longrightarrow$Cauchy.

**Complete Space** - to get the converse Cauchy$\Longrightarrow$Convergence.

A metric space $S$ is complete if and only if every Cauchy sequence converges to a point in $S$.

**Open and Closed**

A set $K$ is open in a metric space if for all $x \in K$ there exists $\varepsilon(x) > 0$ s.t. the ball $\{y : \rho(x, y) < \varepsilon\}$ is contained in $K$

$\overline{L}$, The closure of a set $L$ consists of all points $x$ such that every ball at $x$ contains a point of $L$

A set $L$ is closed if it is equal to its closure, $L = \overline{L}$.

The complement of a closed set is open and vice versa.

$L$ is closed if and only if every convergent sequence in $L$ converges to a point in $L$.

**Continuity**

A function $T : X \to Y$ where $X$ and $Y$ are metric spaces is continuous at $x \Leftrightarrow$ every $\varepsilon > 0$ there is a $\delta > 0$ s.t. $\rho(T(x), T(y)) < \varepsilon$ for every $y$ that satisfies $\rho(x, y) < \delta$

Equivalent definition: $T$ is continuous at $x \Longleftrightarrow$ for any sequence $x_n \to x$, $T(x_n) \to T(x)$

**Compactness** - useful for maximum theorem

A set $K$ in a metric space is compact $\Leftrightarrow$every sequence $\{x_n\}$ in $K$ has a convergent subsequence.

In Euclidean space compactness is equivalent to closed and bounded. Compact spaces are complete.

**Contractions** - useful for constructing solutions to fixed point problems

$T : S \to S$, a function on a metric space, is a contraction $\Leftrightarrow \exists \beta \in (0, 1)$ with $\rho(Tx, Ty) \leq \beta\rho(x, y)$

**Contraction Mapping Theorem**

If $T$ is a contraction on a complete metric space, it has a unique fixed point $v$ and $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$

**Blackwell's Sufficient Conditions**

$T$, a bounded function is a contraction if:

i) $f \leq g \Longrightarrow Tf \leq Tg$ ii) $T(f + a)(x) \leq (Tf)(x) + \beta a$