

Problem Set 6 Answers

Question A1

$$\begin{aligned} E[X^2] &= \int_0^1 x^2 f(x) dx \\ &= \int_0^1 x^2 dx \quad \text{since } f(x) = 1 \text{ for } X \sim U[0, 1] \\ &= \left. \frac{x^3}{3} \right|_{x=0}^{x=1} \\ &= \frac{1}{3}. \end{aligned}$$

Question A2

$$\begin{aligned} E[Y] &= \int_0^K x f(x) dx + \int_K^1 0 f(x) dx \\ &= \int_0^K x dx \\ &= \left. \frac{x^2}{2} \right|_{x=0}^{x=K} \\ &= \frac{K^2}{2}. \end{aligned}$$

Question B1

The key to the problem is to write down the correct objective function (i.e., the correct social welfare function). The objective function from class was

$$\begin{aligned} &\int_0^{df} 0 db + \int_{df}^1 (b - h) db - c(d) \\ &= \int_{df}^1 (b - h) db - c(d). \end{aligned}$$

In the extension under consideration in this problem, three changes need to be made. First, we can set $c(d) = 0$ because of the simplifying assumption. Second, we can substitute the fixed value d_0 for d . Third, we need to add the expected cost of punishment. This only is experienced if the crime is committed (i.e., if $b > d_0 f$). Even then, it is only experience with probability d_0 (i.e., the probability the criminal is caught). Thus, we need to add $d_0 \lambda f$,

the expected cost of punishment conditional on a crime being committed, to the integrand. Putting all these changes together, we have

$$\begin{aligned} & \int_{d_0 f}^1 (b - h - d_0 \lambda f) db \\ &= \left(\frac{b^2}{2} - hb - d_0 \lambda f b \right) \Big|_{b=d_0 f}^{b=1} \\ &= \frac{1}{2} - h - d_0 \lambda f - \frac{(d_0 f)^2}{2} + h d_0 f + \lambda (d_0 f)^2. \end{aligned}$$

The first order condition for the maximization of this objective function with respect to f is

$$-d_0 \lambda - d_0^2 f + h d_0 + 2 \lambda d_0^2 f = 0.$$

Now for some technical notes about the second order condition which are beyond the scope of what I want you to know for class, but which are included here for completeness. Differentiating the first order condition with respect to f gives the second order condition

$$-d_0^2 + 2 \lambda d_0^2 = d_0^2 (2 \lambda - 1).$$

The objective function is therefore concave if $\lambda < 1/2$ and convex if $\lambda > 1/2$. If the objective function is convex, the solution for the maximum will be a corner solution. It turns out that the corner solution is $f^* = 0$ if $h < 1/2$ and $f^* = 1/d_0$ (or actually any value greater than this works, too; they all deter the crime completely) if $h > 1/2$.

Even if the objective function is concave, the solution will be a corner solution for λ close to but below $1/2$.

So suppose λ is sufficiently less than $1/2$ so that we have an interior solution. Solving the first order condition, the interior solution is

$$f^* = \frac{1}{2d_0} \left(\frac{h - \lambda}{1/2 - \lambda} \right).$$

We will use this expression in the next subquestion.

Question B2

Compute the derivative

$$\frac{\partial f^*}{\partial \lambda} = \frac{1}{2d_0} \left(\frac{h - 1/2}{(1/2 - \lambda)^2} \right).$$

The right-hand side has the same sign as $h - 1/2$. If $h > 1/2$, therefore, f^* is increasing in λ . If $h < 1/2$, f^* is decreasing in λ . The two relevant effects of an increase in λ are that (i) it makes crime more costly so society may increase punishment to increase deterrence, and (ii) it makes punishment more costly so society may ask for less punishment. The balance of these two factors hinges on the level of h in this example.