14.281. Problem Set 2. Solutions
Note: Many of the solutions are from the earlier TAs.
Problem 1: Efficiency wages vs. subjective bonuses.

Part 1. The following trigger strategies constitute a perfect public equilibrium are for the efficiency wage game.

(1) as long as the agent is not fired the principal offers wage $w = U_0 + c + \frac{r}{1-p}c$

(2) the agent accepts employement iff the wage is larger than U_0

(4) the agent exerts high effort

[Note that as part of the description we already assumed in (6) that if the output is low the agent is fired, otherwise the relation continues].

With those strategies, both high effort and thus high output are achieved in every period. Let's use the one-deviation principle to check that those strategies are indeed in public perfect equilibrium.

Assuming that the Principal follows the strategy, the agent will indeed be willing to accept employment iff $w \geq U_0$. She will be also induced to work hard. If the agent deviates in one period and shirks, she will save costs c this period; with probability p the output is still high and she is not caught, in which case her payoff in all subsequent periods is the same as if she did not deviate; however, with probability $(1-p)$ she is caught and fired, in which case she loses her efficiency premium (which is $w - U_0 - c = \frac{r}{1-p}c$) thereafter, which discounts to $\frac{1}{r}\frac{r}{1-p}c = \frac{1}{1-p}c$. Therefore at this level of w the worker is just indifferent between working hard (incurring costs c), and shirking (and with probability $(1-p)$ losing discounted premium of $\frac{1}{1-p}c$. At any lower wage she will shirk.

Consider now the principal incentives. There is no reason to deviate to paying higher wage. First note, that by (*) we have $y - w = y - U_0 - c - \frac{r}{1-p}c$ ≥ 0 and thus the principal have a weakly positive surplus from the employement relation. Hence the principal doesn't want to deviate to $w < U_0$ as then the agent will not accept the employment at the current period, and the principal will obtain 0. If the principal offers $w \in [U_0, U_0 + c + \frac{r}{1-p}c)$ then the agent accepts employement and shirks. Hence payoff to the principal is

$$py - w + \frac{p}{1+r}\pi$$

where π is the continuation value of the relationship

$$\pi = \frac{1+r}{r}\left(y - U_0 - c - \frac{r}{1-p}c\right) \geq 0.$$

The deviation payoff is best if $w = U_0$. It is lower than π because of the assumption that $U_0 > py$, that is

$$py - U_0 + \frac{p}{1+r}\pi < \frac{p}{1+r}\pi \leq \pi.$$

Part 2. As in Part 1 let us assume that $U_0 > py$. Then same PPE equilibrium as in Part 1 obtains, or more precisely, the following strategies are in PPE.

Let me formulate the trigger strategies separately for the on-path condition (output was always high in the past and there was no public deviation) ond off-path condition (output was low at least once in the past or there was a public deviation):

On-path:

(1a) the principal offers $s = U_0 + c + \frac{r}{1-p}c$ and $b = 0$.

(2a) the agent accepts employement

(4a) the agent exerts high effort

(6a) the principal pays the bonus

Off-path

(1b) the principal offers $s = 0$ and $b = 0$

(2b) the agent accepts employement iff salary is larger than U_0

(4b) the agent shirks

(6a) the principal doesn't pay the bonus

The argument is identical as in Part 1. QED

Remark. This is not the only PPE possible. As above, let me formulate the trigger strategies separately for the on-path condition (output was always high in the past and there was no public deviation) ond off-path condition (output was low at least once in the past or there was a public deviation):

On-path:

(1a) the principal offers $s = U_0 + c - \frac{c}{1-p}$ and $b = \frac{c}{1-p}$.

(2a) the agent accepts employement

(4a) the agent exerts high effort

(6a) the principal pays the bonus

Off-path

(1b) the principal offers $s = 0$ and $b = 0$

(2b) the agent accepts employement iff salary is larger than U_0

(4b) the agent shirks

(6a) the principal doesn't pay the bonus

Note that the on-path payoff to the principal is

$$\pi = \frac{1+r}{r}\left(y - U_0 - c\right)$$

and is positive by assumption (*). To show that above strategies constitute a perfect public equilibrium for the salary-bonus game, first note, that off-path the equilibirum is self-enforcing. Now consider on-path play node.

(1a) Any deviation would be public, and thus the game would enter the off-path mode. Thus the bonus would become irrelevant, and there are effectively two deviation to consider $s = 0$ (and hence 0 payoff thereafter) or $s = U_0$ and hence payoff $py - U_0 < 0$. As $\pi > 0$ so the devaitions are not tempting.

(2a) The agent payoff on equilibrium path is $\frac{1}{r}U_0$ and equals his reservation payoff. Hence the agent is indifferent.

(4a) High effort leads to expected payoff $\frac{1}{r}U_0$ as before. Low effort saves cost c but introduces probability $(1-p)$ of loosing bonus $b = \frac{c}{1-p}$. Hence the agent is indifferent.

(6a) Shirking leads to continuation payoff 0 while paying the bonus leads to continuation payoff

$$-\frac{c}{1-p}+\frac{1}{1+r}\pi=-\frac{c}{1-p}+\frac{1}{r}\left(y-U_0-c\right)$$

which is nonnegative under assumption (*).

Problem 2. Stationary relational contracts

A proof is offered in the appendix of Levine AER 2003. Let me offer a dynamic programming proof (that uses ideas I learned from Ivan Werning's proof of Luis Rayo (2003) result). For the moral hazard case case the following notation is relevant:
$\bar{u}$ is the agent's reservation utility,
$\bar{\pi}$ is the prinicpal's outside option
$e_t \in [0, \bar{e}]$ is agent's effort
$c(e_t)$ is agent's cost
w_t, b_t are salary and wage, respectively
y_t is principal's benefit
and δ is the discount factor.
Let $U(u)$ be the highest utility attainable for the principal given the promised utility level $u \in [\bar{u}, \infty)$ for the agent. For simplicity let us skip the subscript when refering to period t varaible, and use $'$ to denote period $t+1$ variable.
The principal maximizes

$$U(u) = \max_{w,e,b(\cdot),u'(\cdot)} \{-w + E[y - b(y) + \delta U(u'(y)) | e]\}$$

subject to the promise keeping constraint

$$u = w - c(e) + E[b(y) + \delta u'(y) | e] \text{ for all } y$$

the principal's bonus payment constraint

$$-b(y) + \delta U(u'(y)) \geq \delta \bar{\pi} \text{ for all } y \text{ such that } b(y) > 0$$

the agent's bonus payment constraint

$$b(y) + \delta u'(y) \geq \delta \bar{u} \text{ for all } y \text{ such that } b(y) < 0$$

the ex-ante incentive constraints for effort,

$$-c(e) + E[b(y) + \delta u'(y) | e] \geq \max_{\hat{e}_i} \{-c(\hat{e}) + E[b(y) + \delta u'(y) | \hat{e}]\}$$

and finally the participation constraints:

$$u'(y) \geq \bar{u}$$

and

$$U[u'(y)] \geq \bar{\pi}.$$

The problem asks us to show that without loss of optimality $u'(\cdot)$ can be taken to be a constant. An inspection of the problem shows that this is equivalent to showing that $S(u) \equiv u + U(u)$ is a constant. Let me proceeds in several simple steps.

Step 1. Note that maximizing $U(u)$ given u is equivalent to maximizing $S(u) \equiv u + U(u)$. Rewrite the above program in terms of $S(u)$ as:

$$S(u) = \max_{w,e,b(\cdot),u'(\cdot)} u + \{-w + E[y - b(y) + \delta S(u'(y)) - \delta u'(y)|e]\}$$

subject to

$$u = w - c(e) + E[b(y) + \delta u'(y)|e]$$

$$-b(y) + \delta S(u'(y)) - \delta u'(y) \geq \delta \bar{\pi} \text{ for all } y \text{ such that } b(y) > 0$$

$$b(y) + \delta u'(y) \geq \delta \bar{u} \text{ for all } y \text{ such that } b(y) < 0$$

$$-c(e) + E[b(y) + \delta u'(y)|e] \geq \max_{\hat{e}_i} \{-c(\hat{e}) + E[b(y) + \delta u'(y)|\hat{e}]\}$$

$$u'(y) \geq \bar{u}$$

$$S[u'(y)] - u'(y) \geq \bar{\pi}.$$

Step 2a: Let's substitute for u into the objective from the first constraint (promised agent's utility). Then u appears in the problem only in that constraint hence we can drop that constraint. We can then also drop the wage variable w.
Step 2b: After Step 2a, u no longer appears in the constraint nor objective, so $S(u) = \bar{S}$ for some constant $\bar{S}$. Hence we can further transform the program to

$$S(u) = \bar{S} = \max_{e,b(\cdot),u'(\cdot)} \{-c(e) + E[y + \delta S(u'(y))|e]\}$$

subject to

$$-b(y) + \delta S(u'(y)) - \delta u'(y) \geq \delta \bar{\pi} \text{ for all } y \text{ such that } b(y) > 0$$

$$b(y) + \delta u'(y) \geq \delta \bar{u} \text{ for all } y \text{ such that } b(y) < 0$$

$$-c(e) + E[b(y) + \delta u'(y)|e] \geq \max_{\hat{e}_i} \{-c(\hat{e}) + E[b(y) + \delta u'(y)|\hat{e}]\}$$

$$u'(y) \geq \bar{u}$$

$$S[u'(y)] - u'(y) \geq \bar{\pi}.$$

Step 3: Note that the environment is stationary, hence same argument about $S(u') = \bar{S}$ can be made at time $t+1$. Substituting into the objective and the first and last constraints we obtain

$$\bar{S} = \max_{e,b(\cdot),u'(\cdot)} \left\{-c(e) + E[y|e] + \delta \bar{S}\right\}$$

subject to

$$-b(y) + \delta \bar{S} - \delta u'(y) \geq \delta \bar{\pi} \text{ for all } y \text{ such that } b(y) > 0$$

$$b(y) + \delta u'(y) \geq \delta \bar{u} \text{ for all } y \text{ such that } b(y) < 0$$

$$-c(e) + E[b(y) + \delta u'(y)|e] \geq \max_{\hat{e}_i} \{-c(\hat{e}) + E[b(y) + \delta u'(y)|\hat{e}]\}$$

$$u'(y) \geq \bar{u}$$
$$\bar{S} - u'(y) \geq \bar{\pi}.$$

Step 4: Without loss in optimality we can take $u'(y) = \bar{u}$. That is, if $e, b(\cdot), u'(\cdot)$ satisfy the constraints then $\tilde{e} = e, \tilde{b}(y) = b(y) + \delta[u'(y) - \bar{u}]$, $\tilde{u}'(y) = \bar{u}$, satisfy all the constraints as well. Hence the claim is proved.

As a bonus note that Step 4 allows us to drop the constraint $u'(y) \geq \bar{u}$ and substitute for $u'(y)$ into remaining constraints. After we do this the second constraint simplifies to $b(y) \geq 0$. After dropping $u'(\cdot)$ from the program this leads us to the following static problem (indexed by $\bar{S}$)

$$\max_{e,b(\cdot)} \left\{-c(e) + E[y|e] + \delta\bar{S}\right\}$$

subject to

$$-b(y) + \delta\bar{S} - \delta\bar{u} \geq \delta\bar{\pi} \text{ for all } y \text{ such that } b(y) > 0$$
$$b(y) \geq 0$$
$$-c(e) + E[b(y) + \delta\bar{u}|e] \geq \max_{\hat{e}_i}\{-c(\hat{e}) + E[b(y) + \delta\bar{u}|\hat{e}]\}$$
$$\bar{S} \geq \bar{u} + \bar{\pi}.$$

As noted already, this problem is static and thus, moving backwards, the solution to the original problem is stationary. Note that we can now solve for $\bar{S}$ as the fixed point of the above problem.

Problem 3: Relational Contract Meets Mulitask

(a) $\underset{a_1,a_2}{Max} y_L + (f_1a_1 + f_2a_2)(y_H - y_L) - c(a_1, a_2)$

FOC gives that $a_i^{FB} = f_i(y_H - y_L)$.

(b) This part is almost identical to the 1st problem. On the equilibrium path, (1) the principal offers the agent s, (2) the agent chooses the optimal action, (3) the principal offers the agent bonus b if y_H occurs. If either (1) or (3) fails, the parties fall back to the outside options forever.

The IR and IC for the Principal is

PIC1(which indirectly implies PIR)

$$y_L - s + \frac{1}{r}(y_L - s) \geq y_L + \frac{1}{r}\pi_0$$

PIC2

$$-b + \frac{1}{r}(y_L - s) \geq \frac{1}{r}\pi_0$$

The IR and IC for the Agent is

AIC

$$b = y_H - y_L$$

AIR

$$s + b(f_1a_1^{FB} + f_2a_2^{FB}) - C^{FB} \geq U_0$$

The usual Levin Condition for FB requires

$$b \leq \frac{1}{r}(V^{FB} - V_0)$$

In other words, we need

$$r^* \leq \frac{V^{FB} - V_0}{y_H - y_L}$$

(c) This is a standard multi-task problem, and we need to maximize the total surplus subject to the agent's IC constraint.

The agent

$$\underset{a_1,a_2}{Max} s + (g_1a_1 + g_2a_2)b - c(a_1, a_2)$$

and we get

$$a_i = g_ib$$

Now the Principal

$$\underset{b}{Max} y_L + (f_1g_1 + f_2g_2)b(y_H - y_L) - c(g_1b, g_2b)$$

We get the standard solution that

$$b^* = \frac{f_1 g_1 + f_2 g_2}{g_1^2 + g_2^2}(y_H - y_L)$$

(d) This is almost the same as in (b). The only difference is that the old outside options(U_0, π_0) are now replaced by $EU(s_a, b^*)$ and $E\pi(s_a, b^*)$.

In particular, the most relevant IC for the principal and IC and IR for the agent gives

PIC:

$$-b + \frac{1}{r}(y_L - s) \geq \frac{1}{r}E\pi(s_a, b^*)$$

AIC:

$$b = y_H - y_L$$

AIR:

$$s + b(f_1 a_1^{FB} + f_2 a_2^{FB}) - C^{FB} \geq EU(s_a, b^*)$$

Therefore, for the first best to be achieved here, we need

$$r^* \leq \frac{V^{FB} - V^{MT}}{y_H - y_L} < \frac{V^{FB} - V_0}{y_H - y_L}$$

where we have

$$V^{MT} = EU(s_a, b^*) + E\pi(s_a, b^*)$$