14.281 Problem Set 3. Solutions
Some solutions are from previous TAs.

Problem 1: Normal learning model.

Let us first prove

Lemma. Suppose that (η, ε) is bivariate normal with marginal distributions $\eta \sim N(m, \frac{1}{h_0})$, $\varepsilon \sim N(0, \frac{1}{h_\varepsilon})$. Denote $z_1 = \eta + \varepsilon$. Then the conditional distribution of $\eta|z_1$ is $N(\frac{h_0}{h_0+h_\varepsilon}m_0 + \frac{h_\varepsilon}{h_0+h_\varepsilon}z_1, \frac{1}{h_0+h_\varepsilon})$.

Proof. Clearly $z_1 \sim N(m_0, \frac{1}{h_0} + \frac{1}{h_\varepsilon})$, and $z_1|\eta \sim N(\eta, \frac{1}{h_\varepsilon})$. By Bayes rule

$$f(\eta|z_1) = \frac{f(z_1,\eta)}{f(z_1)} = \frac{f(z_1|\eta)f(\eta)}{f(z_1)} = \frac{\frac{1}{\sqrt{2\pi}}\sqrt{h_\varepsilon}e^{-\frac{h_\varepsilon}{2}(z_1-\eta)^2} \cdot \frac{1}{\sqrt{2\pi}}\sqrt{h_*}e^{-\frac{h_*}{2}(z_1-m_*)^2}}{\frac{1}{\sqrt{2\pi}}\sqrt{\frac{h_0h_\varepsilon}{h_0+h_\varepsilon}}e^{-\frac{1}{2}(z_1-m_0)^2\frac{h_0h_\varepsilon}{h_0+h_\varepsilon}}}$$

$$= \frac{1}{\sqrt{2\pi}}\sqrt{h_0+h_\varepsilon}\sqrt{\frac{h_*}{h_0}}e^{-\frac{h_0+h_\varepsilon}{2}\left[\eta-\left(\frac{h_\varepsilon z_1}{h_0+h_\varepsilon}+\frac{m_0h_0}{h_0+h_\varepsilon}\right)\right]^2},$$

meaning that $\eta|z_1 \sim N(\frac{h_0}{h_0+h_\varepsilon}m_0 + \frac{h_\varepsilon}{h_0+h_\varepsilon}z_1, \frac{1}{h_0+h_\varepsilon})$ which proves the lemma. QED

Now the first claim of the problem immediately follow from the lemma.

Finally, let me show the second claim for $t = 2$. Note that $\eta|z_1$ and ε_2 are independent and normal and hence $((\eta|z_1), \varepsilon_2)$ is bivariate normal. Moreover

$$\begin{aligned}(\eta|z_1, z_2) &= (\eta|z_2)|z_1 = ((\eta|(z_2|z_1)))|z_1 = ((\eta|((\eta+\varepsilon_2)|z_1)))|z_1 \\ &= ((\eta|((\eta|z_1)+\varepsilon_2)))|z_1 = (\eta|((\eta|z_1)+\varepsilon_2), z_1) = (\eta|z_1)|((\eta|z_1)+\varepsilon_2).\end{aligned}$$

Thus we can apply our result for $t = 1$ and the Lemma to conclude that $(\eta|z_1, z_2) = (\eta|z_1)|((\eta|z_1)+\varepsilon_2)$ is normally distributed with precision $h_2 = h_0 + 2h_\varepsilon$ and mean

$$\left(\frac{h_0+h_\varepsilon}{h_0+2h_\varepsilon}\left(\frac{h_0}{h_0+h_\varepsilon}m_0 + \frac{h_\varepsilon}{h_0+h_\varepsilon}z_1\right) + \frac{h_\varepsilon}{h_0+2h_\varepsilon}((\eta|z_1)+\varepsilon_2)\right)|(z_1, z_2)$$
$$= \frac{h_0}{h_0+2h_\varepsilon}m_0 + \frac{h_\varepsilon}{h_0+2h_\varepsilon}z_1 + \frac{h_\varepsilon}{h_0+2h_\varepsilon}z_2.$$

The argument in the general case goes by induction, with the inductive step identical to the above step from $t = 1$ to $t = 2$.

Problem 2: Multi-task meets career concerns

First consider period 2. There are no more career concerns in this period. Given b_2 the agent maximizes

$$\max_{a_{12},a_{22}} b_2E(p_2|a_{12}, a_{22}) - \frac{1}{2}\left(a_{12}^2 + a_{22}^2\right) = b_2(g_1a_{12} + g_2a_{22}) - \frac{1}{2}\left(a_{12}^2 + a_{22}^2\right)$$

and sets

$$a_{12} = b_2 g_1,$$
$$a_{22} = b_2 g_2.$$

Given the perfect competition, the principal wants to maximize the total surplus, and offers b_2 as to maximize

$$\max_{b_2} E\left(y_2 - \frac{1}{2}\left(a_{12}^2 + a_{22}^2\right)|p_1, b_2\right) = E\left(\eta|p_1, y_1\right) + \left(f_1 a_{12} + f_2 a_{22}\right) - \frac{1}{2}\left(a_{12}^2 + a_{22}^2\right)$$

subject to agent's IC given by the above equations. Thus, he maximizes

$$\max_{b_2} \left(f_1 b_2 g_1 + f_2 b_2 g_2\right) - \frac{1}{2} b_2^2 \left(g_1^2 + g_2^2\right)$$

and sets

$$b_2 = \frac{\left(f_1 g_1 + f_2 g_2\right)}{\left(g_1 g_1 + g_2 g_2\right)} = \frac{fg}{gg}.$$

Given that the expected principal's payoff is

$$\begin{aligned}
E\pi_2 = Ey_2 - Ew_2 &= E\left(\eta|p_1, y_1\right) + \left(f_1 a_{12} + f_2 a_{22}\right) - \left(s_2 + b_2 E\left(p_2|p_1, y_1\right)\right) \\
&= E\left(\eta|p_1, y_1\right) + \left(f_1 b_2 g_1 + f_2 b_2 g_2\right) - \left(s_2 + b_2\left(E\left(\eta|p_1, y_1\right) + g_1 b_2 g_1 + g_2 b_2 g_2\right)\right) \\
&= \left(1 - \frac{fg}{gg}\right) E\left(\eta|p_1, y_1\right) - s_2 + \frac{fg}{gg}\left(fg\right) - \left(\frac{fg}{gg}\right)^2 \left(gg\right) \\
&= \left(1 - \frac{fg}{gg}\right) E\left(\eta|p_1, y_1\right) - s_2
\end{aligned}$$

and equals 0 due to the competition among principals. Hence

$$s_2 = \left(1 - \frac{fg}{gg}\right) E\left(\eta|p_1, y_1\right).$$

Note for future reference, that the total surplus in the second period

$$\begin{aligned}
S_2 &= E\left(\eta|p_1, y_1\right) + \left(f_1 a_{12} + f_2 a_{22}\right) - \frac{1}{2}\left(a_{12}^2 + a_{22}^2\right) \\
&= E\left(\eta|p_1, y_1\right) + \frac{fg}{gg}\left(fg\right) - \frac{1}{2}\left(\frac{fg}{gg}\right)^2 \left(gg\right) \\
&= E\left(\eta|p_1, y_1\right) + \frac{1}{2}\frac{\left(fg\right)^2}{gg}
\end{aligned}$$

Now, consider period 1. Given b_1 and the rational expectation of s_2 the agent maximizes

$$\begin{aligned}
&\max_{a_{11}, a_{21}} b_1 E\left(p_1|a_{11}, a_{21}\right) - \frac{1}{2}\left(a_{11}^2 + a_{21}^2\right) + \delta E\left(s_2|p_1, y_1|a_{11}, a_{21}\right) \\
&= b_1\left(g_1 a_{11} + g_2 a_{21}\right) - \frac{1}{2}\left(a_{11}^2 + a_{21}^2\right) + \delta\left(1 - \frac{fg}{gg}\right) E\left(\eta|p_1, y_1|a_{11}, a_{21}\right).
\end{aligned}$$

and sets

$$a_{11} = b_1 g_1 + \delta \left(1 - \frac{fg}{gg}\right) \frac{\partial}{\partial a_{11}} E\left(\eta | p_1, y_1 | a_{11}, a_{21}\right),$$
$$a_{21} = b_1 g_2 + \delta \left(1 - \frac{fg}{gg}\right) \frac{\partial}{\partial a_{11}} E\left(\eta | p_1, y_1 | a_{11}, a_{21}\right).$$

To compute $E\left(\eta | p_1, y_1 | a_{11}, a_{21}\right)$ first note that an argument identical to that of Problem 3 of PS2 shows that

$$E\left(\eta | p_1, y_1\right) = \frac{h}{h + h_\varepsilon + h_\phi} m + \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi}\left(p_1 - g_1 a_{11}^* - g_2 a_{21}^*\right) + \frac{h_\phi}{h + h_\varepsilon + h_\phi}\left(y_1 - f_1 a_{11}^* - f_2 a_{21}^*\right),$$

where a_{11}^*, a_{21}^* are principal's conjectures of a_{11}, a_{21}. Thus,

$$E\left(\eta | p_1 | a_{11}, a_{21}\right)$$
$$= E\left(\frac{h}{h + h_\varepsilon + h_\phi} m + \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi}\left(p_1 - g_1 a_{11}^* - g_2 a_{21}^*\right) + \frac{h_\phi}{h + h_\varepsilon + h_\phi}\left(y_1 - f_1 a_{11}^* - f_2 a_{21}^*\right) | a_{11}, a_{21}\right)$$
$$= \frac{h}{h + h_\varepsilon + h_\phi} m + \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi}\left(m + g_1 a_{11} + g_2 a_{21} - g_1 a_{11}^* - g_2 a_{21}^*\right)$$
$$+ \frac{h_\phi}{h + h_\varepsilon + h_\phi}\left(m + f_1 a_{11} + f_2 a_{21} - f_1 a_{11}^* - f_2 a_{21}^*\right)$$

and consequently

$$\frac{\partial}{\partial a_{11}} E\left(\eta | p_1, y_1 | a_{11}, a_{21}\right) = \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1,$$
$$\frac{\partial}{\partial a_{12}} E\left(\eta | p_1, y_1 | a_{11}, a_{21}\right) = \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2.$$

Plugging this result into formulas on a_{11}, a_{21} we obtain

$$a_{11} = b_1 g_1 + \delta \left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1\right),$$
$$a_{21} = b_1 g_2 + \delta \left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2\right).$$

Note also that in equilibrium $a_{11}^* = a_{11}, a_{21}^* = a_{21}$ and thus

$$E\left(E\left(\eta | p_1, y_1\right) | b_1\right)$$
$$= E\left(\frac{h}{h + h_\varepsilon + h_\phi} m + \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi}\left(p_1 - g_1 a_{11} - g_2 a_{21}\right) + \frac{h_\phi}{h + h_\varepsilon + h_\phi}\left(y_1 - f_1 a_{11} - f_2 a_{21}\right) | b_1\right)$$
$$= \frac{h}{h + h_\varepsilon + h_\phi} m + \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} m + \frac{h_\phi}{h + h_\varepsilon + h_\phi} m = m$$

and thus the second period surplus

$$E\left(S_2 | b_1\right) = m + \frac{1}{2} \frac{\left(fg\right)^2}{gg}$$

does not depend on period 1 decision of the principal.

Given that the principal expects to earn 0 next period and perfect competiton with other principals in the current period, and given above IC constraints the principal offers b_1 to maximize the total surplus in both periods

$$
\begin{aligned}
&\max_{b_1} E\left(y_1 - \frac{1}{2}\left(a_{11}^2 + a_{21}^2\right)|b_1\right) + E\left(S_2|b_1\right) \\
&= \left(f_1 a_{11} + f_2 a_{21}\right) - \frac{1}{2}\left(a_{11}^2 + a_{21}^2\right) + E\left(S_2|b_1\right) \\
&= f_1\left(b_1 g_1 + \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1\right)\right) \\
&+ f_2\left(b_1 g_2 + \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2\right)\right) \\
&- \frac{1}{2}\left(b_1 g_1 + \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1\right)\right)^2 \\
&- \frac{1}{2}\left(b_1 g_2 + \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2\right)\right)^2 \\
&+ E\left(S_2|b_1\right) \\
&= \text{const} + \left(f_1 b_1 g_1 + f_2 b_1 g_2\right) - \frac{1}{2}\left(b_1 g_1\right)^2 - \frac{1}{2}\left(b_1 g_2\right)^2 \\
&- b_1 g_1 \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1\right) \\
&- b_1 g_2 \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2\right)
\end{aligned}
$$

and facing a concave objetcive with FOC:

$$
\begin{aligned}
&b_1\left(g_1 g_1 + g_2 g_2\right) \\
&= \left(f_1 g_1 + f_2 g_2\right) \\
&- \delta\left(1 - \frac{fg}{gg}\right)\left(g_1\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_1 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_1\right) - g_2\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} g_2 + \frac{h_\phi}{h + h_\varepsilon + h_\phi} f_2\right)\right) \\
&= \left(f_1 g_1 + f_2 g_2\right) - \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi}\left(g_1 g_1 + g_2 g_2\right) - \frac{h_\phi}{h + h_\varepsilon + h_\phi}\left(g_1 f_1 + g_2 f_2\right)\right)
\end{aligned}
$$

sets

$$
\begin{aligned}
b_1 &= \frac{fg - \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} gg + \frac{h_\phi}{h + h_\varepsilon + h_\phi} gf\right)}{gg} \\
&= \frac{fg}{gg} - \delta\left(1 - \frac{fg}{gg}\right)\frac{\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} gg + \frac{h_\phi}{h + h_\varepsilon + h_\phi} gf}{gg} \\
&= \frac{fg}{gg} - \delta\left(1 - \frac{fg}{gg}\right)\left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} + \frac{h_\phi}{h + h_\varepsilon + h_\phi}\frac{fg}{gg}\right)
\end{aligned}
$$

Hence the expected principal payoff in first period is

$$E\pi_1 = Ey_1 - Ew_1 = m + (f_1 a_{11} + f_2 a_{21}) - (s_1 + b_1 E(p_1|b_1))$$

The competition among the principals drives the payoff to 0 thus determining

$$s_1 = m + (f_1 a_{11} + f_2 a_{21}) - b_1 (m + g_1 a_{11} + g_2 a_{21})$$

which can be straightforwardly computed using above formulas for a_{11}, a_{21} and b_1.

Finally, let us compare

$$b_2 = \frac{fg}{gg} = \cos(f,g) \frac{|f|}{|g|} \in R$$

and

$$\begin{aligned} b_1 &= \frac{fg}{gg} - \delta \left(1 - \frac{fg}{gg}\right) \left(\frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} + \frac{h_\phi}{h + h_\varepsilon + h_\phi} \frac{fg}{gg}\right) \\ &= b_2 - \delta (1 - b_2) \frac{h_\varepsilon}{h + h_\varepsilon + h_\phi} - \delta (1 - b_2) \frac{h_\phi}{h + h_\varepsilon + h_\phi} b_2. \end{aligned}$$

Note that, in general, it is ambiguous whether b_1 or b_2 is larger. The difference between the two incentive slopes is a consequence of the "free" incentives or disincentives provided by the career concerns. Roughly speaking, the first term in the difference corresponds to the information transmitted via y, while the second term corresponds to the information transmitted via p (this only rough categorization as the two signals interact).

Problem 3: Influence Activities

(a): Let the equilibrium action be a_1^* and a_2^*. Then in stage (iv), party i tries to solve

$$\underset{d}{Max} - k_i E[(d - s - b_i)^2 | \sigma, a_i^*, a_j^*]$$

(b) Let us denote

$$y = d - s - b_i | \sigma, a_i^*, a_j^*$$

It's easy to see that y is distrbuted as

$$N(d - b_i - (\frac{mh + (\sigma - a_i^* - a_j^*)h_\varepsilon}{h + h_\varepsilon}), h + h_\varepsilon)$$

Now

$$E[(d - s - b_i)^2 | \sigma, a_i^*, a_j^*] = E(y^2) = [d - b_i - (\frac{mh + (\sigma - a_i^* - a_j^*)h_\varepsilon}{h + h_\varepsilon})]^2 + \frac{1}{h + h_\varepsilon}$$

Therefore, the decision

$$d^* = b_i + \frac{mh + (\sigma - a_i^* - a_j^*)h_\varepsilon}{h + h_\varepsilon}$$

(c) Given the optimal decision d*, agent j chooses in stage (ii) to

$$\underset{a_j}{Max} - k_j E[(d^* - s - b_j)^2 | \sigma, a_j] - c(a_j)$$

(d) As in (b), if we denote

$$x = d^* - s - b_j | \sigma, a_j$$

we have that x is distributed as

$$N(b_i - b_j + \frac{h_\varepsilon (a_j - a_j^*)}{h + h_\varepsilon}, h + h_\varepsilon)$$

Therefore,

$$E[(d^* - s - b_j)^2 | \sigma, a_j] = (b_i - b_j + \frac{h_\varepsilon (a_j - a_j^*)}{h + h_\varepsilon})^2 + \frac{1}{h + h_\varepsilon}$$

If we look at the FOC of agent j's problem, we get

$$-2k_j \frac{h_\varepsilon}{h + h_\varepsilon}(b_i - b_j) = c'(a_j^*)$$

(e) From the last expression in (d), we see the the magnitude of the influence activity increases with k_j.

(f) Clearly $a_i^* = 0$. Therefore, the expected loss of agent i is

$$\frac{k_i}{h + h_\varepsilon}$$

Similarly, the expected loss of agent j is

$$\frac{k_j}{h + h_\varepsilon} + k_j(b_i - b_j)^2 + c(a_j^*)$$

we get the total loss by summing the above two expressions.

Problem 4:

(a): In period 2, for a given rent r_2, the tenants will

$$\underset{e_2}{Max} p - r_2 - E[\beta - e_2 - \varepsilon_2] - g(e_2)$$

As a result, we have

$$g'(e_2^*) = 1$$

Since the tenant earns a zero expected income in period 2, we have

$$r_2 = p - \widehat{\beta(c_1)} + e_2^* - g(e_2^*)$$

where $\widehat{\beta(c_1)}$ is the estimated cost charactristic and e_2^* is the equilibrium second period effort.

In period 1, the initial rent satisfies

$$r_1 = p + b_0 - e_1^*$$

The tenant

$$\underset{e_1}{Max} p - r_1 - E[\beta - e_1 - \varepsilon_1] - g(e_1) + E[\widehat{\beta(c_1)}|e_1]$$

For Nash Equilibrim. we need also

$$e_1^* \in \arg\max_{e_1} p - r_1 - E[\beta - e_1 - \varepsilon_1] - g(e_1) + E[\widehat{\beta(c_1)}|e_1]$$

(b) Notice that in equilibrium the entrepreneur believes that

$$c_1 = \beta - e_1^* - \varepsilon_1$$

Therefore, the entrepreneur's update on cost parameter given observed cost is

$$\widehat{\beta(c_1)} = \frac{h_0(b_0 - e_1^*) + h_\varepsilon c_1}{h_\varepsilon + h_0}$$

If the tenant chooses effort e_1,the expected belief on the cost parameter becomes

$$E[\widehat{\beta(c_1)}|e_1] = \frac{h_0(b_0 - e_1^*) + h_\varepsilon((b_0 - e_1)}{h_\varepsilon + h_0}$$

Given above, the FOC of the tenant's maximization problem in period 1 is

$$g'(e_1^*) = \frac{h_0}{h_\varepsilon + h_0}$$

Therefore, we have

$$e_1^* < e_2^*$$

(c) In contrast to the standard career concern model, the effort in this problem is lower in the first period. This results from the fact that the estimate for the production characteristic is increased by lowering, not increasing the efforts.