14.281 Problem Set 4. Solutions

Some Answers are from Previous TAs.

Problem 1. Production and machine maitenance

Note that as usual in the multitasking framework, we assume that mean of ε and φ is 0. Assume also that agent's reservation utility is 0.

(a) Given b the agent maximizes

$$\max_{a_1,a_2} bE\left(p|a_1,a_2\right)-\frac{1}{2}\left(a_1^2+a_2^2\right)=b\left(g_1a_1+g_2a_2\right)-\frac{1}{2}\left(a_1^2+a_2^2\right)$$

and sets

$$\begin{aligned}a_1&=bg_1,\\ a_2&=bg_2.\end{aligned}$$

The principal wants to maximize the total surplus, and offers b as to maximize

$$\max_b E\left(y+v-\frac{1}{2}\left(a_1^2+a_2^2\right)|b\right)=\left(\left(f_1+h_1\right)a_1+\left(f_2+h_2\right)a_2\right)-\frac{1}{2}\left(a_1^2+a_2^2\right)$$

subject to agent's IC given by the above equations. Thus, he maximizes

$$\begin{aligned}&\max_b\left(f_1+h_1\right)bg_1+\left(f_2+h_2\right)bg_2-\frac{1}{2}b^2\left(g_1^2+g_2^2\right)\\ &=b\left(f+h\right)g-\frac{1}{2}b^2gg\end{aligned}$$

and sets

$$b_E^*=b=\frac{\left(f+h\right)g}{gg}.$$

Given that the expected total surplus S_E is

$$\begin{aligned}S_E&=b\left(f+h\right)g-\frac{1}{2}b^2gg\\ &=\frac{\left(f+h\right)g}{gg}\left(f+h\right)g-\frac{1}{2}\left(\frac{\left(f+h\right)g}{gg}\right)^2gg\\ &=\frac{1}{2}\frac{\left[\left(f+h\right)g\right]^2}{gg}\end{aligned}$$

(b) Now, proceeding as above we obtain that the agent sets

$$\begin{aligned}a_1&=bg_1+h_1,\\ a_2&=bg_2+h_2,\end{aligned}$$

and the principal wants to maximize the total surplus, and offers b as to maximize

$$\max_b E\left(y+v-\frac{1}{2}\left(a_1^2+a_2^2\right)|b\right)=\left(f+h\right)a-\frac{1}{2}aa$$

subject to agent's IC given by the above equations. Thus, he maximizes

$$\begin{aligned}
&\max_b (f+h)(bg+h) - \frac{1}{2}(bg+h)(bg+h) \\
&= (f+h)bg - \frac{1}{2}\left(b^2 gg + 2bhg\right) + \text{a constant independent of } b
\end{aligned}$$

and sets

$$b_I^* = b = \frac{(f+h)g - hg}{gg} = \frac{fg}{gg}.$$

Given that the expected total surplus S_E is

$$\begin{aligned}
S_I &= (f+h)a - \frac{1}{2}aa \\
&= (f+h)\left(\frac{fg}{gg}g + h\right) - \frac{1}{2}\left(\frac{fg}{gg}g + h\right)^2 \\
&= \frac{1}{2}\frac{(fg)^2}{gg} + \frac{1}{2}hh + fh + \frac{fg}{gg}hg - \frac{fg}{gg}hg \\
&= \frac{1}{2}\frac{(fg)^2}{gg} + \frac{1}{2}hh + fh
\end{aligned}$$

(c) Note that the first best level of effort (maximizing total surplus) is $a = f + h$. Consequently, if $g = f + h$ (or, more generally, $g \parallel f + h$), S_E is first best and hence $S_E > S_I$ as long as $h \neq 0$.

In general, the condition is

$$\frac{1}{2}\frac{\left[(f+h)g\right]^2}{gg} > \frac{1}{2}\frac{(fg)^2}{gg} + \frac{1}{2}hh + fh$$

which very roughly speaking means that p is a good signal for $v + y$ compared to f and that production and machine maintenance are competing activities.

(d) Likewise, S_I is first best if you can make $h + bg = f + h$ which you can do when $f \parallel g$. Unless $f \parallel h$, that will guarantee $S_I > S_E$.

More generally, as in (c) the condition is

$$\frac{1}{2}\frac{\left[(f+h)g\right]^2}{gg} < \frac{1}{2}\frac{(fg)^2}{gg} + \frac{1}{2}hh + fh$$

which roughly speaking means that p is a good signal for y compared to $y + v$ and that f and h (production and machine maintenance) are complimentary.

Problem 2.

(a) The first best decision rule is to implement decision $d = d_{A_s}$ in state s.

(b) If the agent is given the decision right, he will choose most efficient decision ex post and no bargaining is necessary. Therefore, the agent

$$\max_{a_1,a_2} h_1 a_1 + h_2 a_2 - \frac{1}{2}\left(a_1^2 + a_2^2\right)$$

Therefore, we have

$$a_1^* = h_1, a_2^* = h_2$$

The total suplus is

$$h_1 g_1 + h_2 g_2 + \frac{1}{2}\left(h_1^2 + h_2^2\right) + K_A$$

(c) If the principal has the decision right, without bargaining with the agent he will choose $d = d_{A_p}$, which is inefficient. Therefore, Nash Bargaining will occur. Since the agent's outside option is zero, his payoff after the bargaining is

$$\frac{1}{2}[(h_1 + g_1)a_1 + (h_2 + g_2)a_2 + K_A - f_1 a_1 - f_2 a_2 - K_P] - \frac{1}{2}\left(a_1^2 + a_2^2\right)$$

To maximize the expression above, the agent will choose

$$a_1^* = g_1 + h_1 - f_1, a_2^* = g_2 + h_2 - f_2$$

The party's total surplus is

$$(h_1+g_1)(g_1+h_1-f_1)+(h_2+g_2)(g_2+h_2-f_2)-\frac{1}{2}[(g_1+h_1-f_1)^2+(g_2+h_2-f_2)^2]$$

(d) It is possible to allocate the decision right to the principal. In this way, it might induce the agent to exert effort levels that are closer to the social optimal one. As an example, if $h_1 = f_1 = h_2 = f_2 = 0, g_1 = g_2 = 1$, then a simple computation shows that it's more better to give the principal decision right ex ante.

Problem 3.

Suppose the equilibrium investment of the buyer is I*, then the seller chooses $p \in [I^* - x, I^* + x]$ to

$$\underset{p\in[I^*-x,I^*+x]}{Max} p\frac{I^* + x - p}{2x}$$

The solution to this is

$$p^* = I^* - x \qquad \text{if}$$

3x<=I*

$$p^* = \tfrac{I^*+x}{2} \qquad \text{if}$$

3x>I*

It is easy to check that $p^* = I^* - x$ cannot be part of the equilibrium. In particular, I*=1 in this case, and p*>=$\frac{2}{3}$.And the total payoff of the buyer will be less than 0.

Now if $p^* = \frac{I^*+x}{2}$, then buy choosing investment I, the trade will occur with probability $\frac{I+x-p^*}{2x}$. The buyer's average payoff is $\frac{I+x-p^*}{2}$. Therefore, the buyer chooses I to

$$\underset{I}{Max}(\frac{I+x-p*}{2x})(\frac{I+x-p*}{2}) - \frac{1}{2}I^2$$

First order condition gives that

$$I^* = \frac{p^* - x}{1 - 2x}$$

Together with $p^* = \frac{I^*+x}{2}$, we get

$$I^* = \frac{x}{4x-1}$$
$$p^* = \frac{2x^2}{4x-1}$$

For this to be an equilibrium, we need first

$$I^* <= 3x$$

so we have

$$x >= \frac{1}{3}$$

we also need

$$(\frac{I^*+x-p*}{2x})(\frac{I^*+x-p*}{2}) - \frac{1}{2}(I^*)^2 > 0$$

and this gives

$$x >= \frac{1}{2}$$

Therefore, for a pure strategy equilibrium to exist, we need $x >= \frac{1}{2}$