Solution to PS 5

Problem 1: Adaptation

(a) The first best decision rule is to choose d=0 in s_1 and d=1 in s_2.

(b) If we allocate the decision right to A, the per period expected total payoff ($E[U_A + U_B]$) is $\frac{1}{2}(1+4) = \frac{5}{2}$. If we allocate the decision right to B, the per period expected total payoff is $\frac{1}{2}(6+2) = 4$. Therefore, the second best allocation of the decision right goes to B.

(c) (i) Note that A owns the decision right at time 0 of period 1. We assume that whoever has the decision right in a period inherits the decision right at the beginning of the next period. Consider the following trigger strategy:

On the equilibrium path:

In each period, (1) A obtains the decision right through bargaining.
(2) B pays A k>0
(3) A chooses d=0 in s_1 and d=1 in s_2.
(4) B pays A m(s)

Off the equilibrium path:

(1) No bargaining occurs. The owner of the decision right keeps the decision right.
(2) No transfers of money take place between parties.
(3) The owner of the decision right chooses a decision favorable to him.
(4) No transfers take place.

(ii) Denote U_A^{FB}, U_B^{FB} as the equilibrium payoff for A and B. $U_A^{FB} + U_B^{FB} = U^{FB}$
Define U_A^{NE}, U_B^{NE} as the off-equilibrium payoff for A and B. $U_A^{NE} + U_B^{NE} = U^{NE}$ For the strategy above to be a SPE, we need to check the following IC constraints.
First, B is willing to give k. This is the same as

$$U_B^{NE} \le U_B^{FB}$$

Second, B is willing to give m(s) to A for all s, i.e.

$$m(s) \le \frac{1}{r}(U_B^{FB} - U_B^{NE})$$

The left hand side is B's current period gain if he fails to pay for m(s). The right hand is the loss of continuation payoff.

Third, A will want to choose d=0 in s_1, i.e.

$$1 - m(s_1) \le \frac{1}{r}(U_A^{FB} - U_A^{NE})$$

The left hand side is A's gain in the current period if he deviates. A receives 1 by choosing d=1 and loses the payment from B $m(s_1)$. The right hand side is the loss of continuation value.

By adding the above inequalities, a necessary condition is

$$1 \leq \frac{1}{r}(U^{FB} - U^{NE}) = \frac{1}{r}(5 - \frac{5}{2})$$

Note that in this equilibrium A has the decision right on the off-equilibrium path, so $U_A^{NE} = \frac{5}{2}, U_B^{NE} = 0$.

Note that the necessary condition is also sufficient. In particular, consider the case that B always pays m to A after the bargaining. And no other transfer is involved. For B to be willing to pay for m, we need,

$$-m + \frac{1}{2}(6) + \frac{1}{r}(-m + \frac{1}{2}(6)) \geq 0$$

where the left hand side is U_B^{FB} and the right hand side is U_B^{NE}. In other words, we need

$$m \leq 3$$

On the other hand, A's incentive constraint is

$$1 + \frac{1}{r}(\frac{1}{2}5) \leq \frac{1}{r}(m + \frac{1}{2}4)$$

By choosing d=0, A receives 0 from the action and a continuation payoff of $\frac{1}{r}(m + \frac{1}{2} * 4)$.

If A chooses d=1, he receives 1 from the action, and her continuation payoff will be $\frac{1}{r}\frac{5}{2}$.

The inequality above can be rewritten as

$$\frac{1}{r}(m - \frac{1}{2}) \geq 1.$$

Let m=3, we see that the maximum value of r is $\frac{5}{2}$.

(iii) If the initial assignment of decision right goes to B, then $U_A^{NE} = 0, U_B^{NE} = 4$. A similar necessary condition as in (ii) says that

$$\text{I} \quad 2 \leq \frac{1}{r}(U^{FB} - U^{NE}) = \frac{1}{r}(5 - 4)$$

where 2 is B's gain from deviation in s_2. Therefore, $r \leq \frac{1}{2}$. As a result, it is better to allocate the initial decision right to A.

Problem 2: Contracting for Control

(a) Let s* be the cutoff state in which A and B have identical payoff:

$$\sigma_B s^* + \rho_B = \sigma_A s^* + \rho_A$$

then the first best rule is to pick

$$d = d_A \qquad \text{if } s<s^*$$
$$d = d_B \qquad \text{if } s>=s^*$$

The expected total payoff is then

$$(s^* - s_L)[\rho_A + \frac{1}{2}\sigma_A(s_L + s^*)] + (s_H - s^*)[\rho_B + \frac{1}{2}\sigma_B(s_H + s^*)]$$

where $s^* = \dfrac{\rho_A - \rho_B}{\sigma_B - \sigma_A}$.

(b) In the spot version, if A controls the decision right, her expected surplus is

$$\rho_A + \frac{1}{2}\sigma_A(s_L + s_H)$$

If B controls the decision right, her expected payoff is

$$\rho_B + \frac{1}{2}\sigma_B(s_L + s_H)$$

Therefore, A controls the decision right if and only if

$$\rho_A - \rho_B \geq \frac{1}{2}(\sigma_B - \sigma_A)(s_L + s_H)$$

(c) Consider the following trigger strategy.
On the equilibrium path:
In each period, (1) A controls the decision right.
(2) A pays B t.
(3) A chooses the first best decision.
(4) A pays B T(d,s)
Off the equilibrium path:
(1) The owner of the decision right keeps the decision right.
(2) No transfers of money take place between parties.
(3) The owner of the decision right chooses a decision favorable to her.
(4) No transfers take place.

For the trigger strategy to be a SPE, there are several possibilities that A will deviate.

(i) Fail to pay t.
(ii) Fail to carry out the correct decision.
(iii) Fail to pay T(d,s)

Denote U_A^{FB}, U_B^{FB} as the equilibrium payoff for A and B. $U_A^{FB} + U_B^{FB} = U^{FB}$ Define U_A^{NE}, U_B^{NE} as the off-equilibrium payoff for A and B. $U_A^{NE} + U_B^{NE} = U^{NE}$.

Then the IC for (i) is like an IR constraint, i.e.

$$U_A^{FB} \geq U_A^{NE}$$

Denote $\Delta_A(s)$ as the extra gain for A from deviation from the first best decision. Then the IC for (ii) is

$$\Delta_A(s) + T(d,s) \le \frac{1}{r}(U_A^{FB} - U_A^{NE})$$

Note that IC in (ii) implies the IC in (iii).($T(d,s) \le \frac{1}{r}(U_A^{FB} - U_A^{NE})$)

Now for B, there are two places he might want to deviate. First, he might fail to pay –t. This is like an IR constraint that requires

$$U_B^{FB} \ge U_B^{NE}$$

Second, he may deviate is failure to pay –T(d,s). The IC for B therefore is

$$-T(d,s) \le \frac{1}{r}(U_B^{FB} - U_B^{NE})$$

Adding the two inequalities above shows that a necessary condition for the trigger strategy to be a SPE is

$$\sup_s \Delta_A(s) \le \frac{1}{r}(U^{FB} - U^{NE})$$

As in problem 1, it is easy to see that this is a sufficient condition as well. To see this, let T(d,s)=0 for all d, s, and let $t = -(s_H - s^*)[\rho_B + \frac{1}{2}\sigma_B(s_H + s^*)]$, so $U_B^{FB} = U_B^{NE} = 0$. It is straightforward to check that A will not deviate in this situation provided

$$\sup_s \Delta_A(s) = \Delta_A(s^*) = \frac{\sigma_B\rho_B - \sigma_A\rho_A}{\sigma_B - \sigma_A} \le \frac{1}{r}(U^{FB} - U^{NE})$$

(d) From the analysis in (c), we see that the maximum discount rate for first best when decision right goes to A is

$$r_A = \frac{U^{FB} - U_A^{NE}}{\sup \Delta_A(s)}$$

When B has the decision right, the maximum discount rate for first best is

$$r_B = \frac{U^{FB} - U_B^{NE}}{\sup \Delta_B(s)}$$

A should be given the decision right if and only if $r_A \ge r_B$. In this case, since $\sup_s \Delta_A(s) = \sup_s \Delta_B(s)$, the maximum temptation to deviate happens at s*, A should have the decision right if and only if $U_A^{NE} \le U_B^{NE}$, i.e.

$$\rho_A - \rho_B \le \frac{1}{2}(\sigma_B - \sigma_A)(s_L + s_H)$$

(e) In this case, the same analysis goes through. We have

$$r_A = \frac{U^{FB} - U_A^{NE}}{\sup \Delta_A(s)} \text{ and } r_B = \frac{U^{FB} - U_B^{NE}}{\sup \Delta_B(s)}$$

And A should be given the decision right if and only if $r_A \ge r_B$.

The only difference is that $\sup_s \Delta_B(s)$ now occurs in s_L instead of s*.

Problem 3: Alliance

(a) To characterize the SPEs of this game, we start backwards. In stage (3), the only action profile that forms a Nash equilibrium of the subgame is

$$m_A = m_B = 0$$

Therefore in stage (2), we must have

$$d_A^*(s) \in \arg\max_{d_A} U_A(d_A, d_B^*(s), s)$$

and

$$d_B^*(s) \in \arg\max_{d_B} U_B(d_B, d_A^*(s), s)$$

(b) (i): The first best decision rule $d^{FB}(s)$ satisfies that

$$(d_A^{FB}(s), d_B^{FB}(s)) \in \arg\max_{d_A, d_B}(U_A(d_A^{FB}(s), d_B^{FB}(s), s) + U_B(d_A^{FB}(s), d_B^{FB}(s), s))$$

(ii): Consider the following trigger strategy:

On the equilibrium path:

(1) Players choose $d^{FB}(s)$

(2) Player A pays B m(s) respectively.

Off the equilibrium path:

(1) Players choose $d^{NE}(s)$

(2) Players pay $m_A = m_B = 0$.

Note that in describing the strategy, we can always assume $m_B = 0$.

Now define the maximum gain from deviation for player i in state s as

$$\Delta_i(s) = \max_{d_i}(U_i(d_i\ (s), d_j^{FB}(s), s) + U_i(d_A^{FB}(s), d_B^{FB}(s), s))$$

And let U_A^{FB}, U_B^{FB} as the equilibrium payoff for A and B. $U_A^{FB} + U_B^{FB} = U^{FB}$ Define U_A^{NE}, U_B^{NE} as the off-equilibrium payoff for A and B. $U_A^{NE} + U_B^{NE} = U^{NE}$

Then as in problem 1 and 2, the necessary conditions become

$$\Delta_A(s) + m(s) \le \frac{1}{r}(U_A^{FB} - U_A^{NE})$$

$$\Delta_B(s) - m(s) \le \frac{1}{r}(U_B^{FB} - U_B^{NE})$$

Then a necessary condition is

$$\sup_s(\Delta_A(s) + \Delta_B(s)) \le \frac{1}{r}(U^{FB} - U^{NE})$$

In general, this is a sufficient condition as well. To see this, denote

$$r = \frac{U^{FB} - U^{NE}}{\sup(\Delta_A(s) + \Delta_B(s))}$$

Now we can choose m(s), for s=1,2,…,n such that

$$r(\Delta_A(s) + m(s)) = (U_A^{FB} - U_A^{NE})$$

Note that U_A^{FB} is a function of m(s) as well.

This can be done in general, because we have n unknowns for n equations.
With these m(s) chosen, we see that

$$\Delta_A(s) + m(s) = \frac{1}{r}(U_A^{FB} - U_A^{NE}) \le \frac{1}{r}(U_A^{FB} - U_A^{NE}) \text{ for all s.}$$

Therefore, the ICs for A are all satisfied.
We now claim that with these m(s) chosen, we must also have

$$r(\Delta_B(s) + m(s)) \le (U_B^{FB} - U_B^{NE})$$

This would implies that the ICs for B is satisfied as well and thus prove the sufficiency.
We prove the claim above by contradiction. If the claim is false, then there exists one state *s'* such that

$$r(\Delta_B(s') + m(s')) > (U_B^{FB} - U_B^{NE})$$

Since

$$r(\Delta_A(s') + m(s')) = (U_A^{FB} - U_A^{NE})$$

Summing the two equations above gives that

$$r(\Delta_A(s') + \Delta_B(s')) > (U^{FB} - U^{NE})$$

In other words,

$$r > \frac{(U^{FB} - U^{NE})}{\Delta_A(s') + \Delta_B(s')} \ge \frac{U^{FB} - U^{NE}}{\sup(\Delta_A(s) + \Delta_B(s))}$$

And this contradicts the definition of r. So we prove the claim and thus prove the sufficiency.
Note that this way of proving the sufficiency is not very constructive because it relies on the existence of solutions for n linear equations.

(c): Trigger Strategy:
On the equilibrium path:

Players choose $d^{FB}(s)$

Off the equilibrium path:

Players choose $d^{NE}(s)$

To determine the larger r, we see that for player A, the ICs are for all s,

$$\Delta_A(s) \le \frac{1}{r}(U_A^{FB} - U_A^{NE})$$

For player B, the ICs are for all s,

$$\Delta_B(s) \le \frac{1}{r}(U_B^{FB} - U_B^{NE})$$

Therefore, the maximum discount

$$r = \underset{i \in \{a,b\}}{Min}\left(\frac{U_i^{FB} - U_i^{NE}}{\sup(\Delta_i(s))}\right)$$