14.381 Final Examination
Fall, 1999

Instructions: This is a closed book exam. You have 180 minutes to answer the questions. Good luck!

1. Let X, Y have a joint distribution and let $M_{x|y}(t)$ be the moment generating function of X conditional on Y.

   a) Show that

   $$M_X(t) = Ee^{tX} = E\left[M_{X|Y}(t)\right]$$

   b) Show that the joint moment generating function of X and Y is given by

   $$M_{X,Y}(t,s) = Ee^{tX+sY} = E\left[e^{sY}M_{X|Y}(t)\right]$$

   c) Assume that conditional on $Y = y$, X has a geometric distribution

   $$\text{Prob}(X = x|Y = y) = y(1-y)^x \text{ for } x = 0, 1, 2 \text{ and } 0 < y < 1$$

   and that $(1 - Y)/Y$ is uniform on $(0, a)$ for $0 < a < \infty$. Show that

   $$M_x(t) = \frac{\ln(1 + a(1 - e^t))}{a(1 - e^t)}.$$

2. Let X and Y have a bivariate density function

   $$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp(-\frac{1}{2}(x^2 - 2\rho xy + y^2)/(1-\rho^2))$$

   a) Find the marginal distribution of Y.

   b) Find the conditional disribution of Y given $X = x$.

   c) What is the distribution of

   $$\frac{(1-\rho)(X+Y)^2}{(1+\rho)(X-Y)^2}.$$

   (Hint: For a) and b) a derivation is required. For c) you can state known facts about the normal and related distributions without proof but make sure to check all the conditions that need to be satisfied for these facts to apply).

3. Let $Y \sim N(0, I_n)$ be an $n \times 1$ vector of multivariate normal random variables with $EY = 0$ and $EYY' = I_n$. Let M be a nonrandom matrix of dimension $n \times n$. Assume M is idempotent and symmetric with $\text{rank}(M) = k$ and $0 < k < n$.

   a) Show that $y'My$ and $y'(I - M)y$ are independent.
   b) Let $t \in \mathbb{R}^n$ be any vector such that $t't < \infty$. Find $Et'My$ and $\text{var}(t'My)$.
   c) Show that $\text{var}(t'y) \geq \text{var}(t'My)$ for all t.
   (Hint: you can use known facts about the normal distribution without proof but make sure to check all the conditions that need to be satisfied for these facts to apply).

4. Let $X_1 ... X_n$ be an iid sample with $EX_i = 0$ and $\text{var}(X_i) = \sigma^2$. Consider the t-ratio

$$t_n = \frac{\sqrt{n}\bar{X}}{\sqrt{S^2}}$$

   where $S^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2$ and $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$.

   a) Find the limiting distribution of t_n as $n \to \infty$.
   b) Find the limiting distribution of $(t_n)^2$ assuming again that $EX_i = 0$. State any theorem you use to prove your answer.
   c) Now assume that $EX_i = \mu > 0$. Show that for any constant M such that $0 < M < \infty$ it follows that $\lim_{n \to \infty} P(t_n > M) = 1$. How do you interpret this result?

5. Let $X_1, ..., X_n$ be an iid sample from an exponential distribution such that X_i has density

$$f_{X_i}(x) = \lambda e^{-\lambda x} \quad \lambda > 0, \ x > 0.$$

   a) Write down the likelihood function for the parameter λ.
   b) Find the ML estimator $\hat{\tau}$ of $\tau(\lambda) = \frac{1}{\lambda}$. Is this estimator unbiased? (Use $EX_i = \frac{1}{\lambda}$).
   c) Show that the ML estimator $\hat{\tau}$ is consistent. Find the asymptotic distribution of $\hat{\tau}$. Verify that all the assumptions in the CLT and WLLN are satisfied (use $\text{var}(X_i) = \frac{1}{\lambda^2}$).
   d) Calculate the Fisher Information of the sample for λ.
   e) Is $\hat{\tau}$ achieving the Cramer-Rao lower bound?