14.381 Midterm Examination

Instructions: This is a closed book exam, but you may refer to one sheet of notes. You have 90 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

1. Let $(X,Y)'$ be a continuous bivariate random vector with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} C\,|xy| & \text{if } x^2+y^2 \le 1, \\ 0 & \text{otherwise,} \end{cases}$$

where C is some positive constant.

(a) Find C.
The constant C is such that $f_{X,Y}$ integrates to one. Therefore,

$$\begin{aligned} C^{-1} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |xy|\,1\left(x^2+y^2\le 1\right)dydx = 4\int_0^{\infty}\int_0^{\infty} xy1\left(x^2+y^2\le 1\right)dydx \\ &= 4\int_0^1\int_0^{\sqrt{1-x^2}} xydydx = 2\int_0^1 x\left(1-x^2\right)dx = 2\left(\frac{1}{2}-\frac{1}{4}\right) = \frac{1}{2}, \end{aligned}$$

where the second equality uses symmetry, the third equality uses $\int_0^{\sqrt{1-x^2}} ydy = \frac{1}{2}\left(1-x^2\right)$, and the fourth equality uses $\int_0^1 x^{k-1}dx = k^{-1}$ (for $k>0$). In other words, $C=2$.

(b) Find $f_X(\cdot)$, the marginal pdf of X.
We have:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy = \int_{-\infty}^{\infty} 2\,|xy|\,1\left(x^2+y^2\le 1\right)dy \\ &= 4\,|x|\int_0^{\infty} y1\left(x^2+y^2\le 1\right)dy. \end{aligned}$$

Clearly, $f_X(x)=0$ if $|x|>1$. If $|x|\le 1$, on the other hand,

$$f_X(x) = 4\,|x|\int_0^{\sqrt{1-x^2}} ydy = 2\,|x|\left(1-x^2\right) = 2\left(|x|-|x|^3\right).$$

Therefore, $f_X(x) = 2\left(|x|-|x|^3\right)1\left(|x|\le 1\right)$.

(c) Find $E(X)$.
Because $f_X(\cdot)$ is even,

$$E(X) = \int_{-\infty}^{\infty} xf_X(x)\,dx = \int_{-1}^{1} xf_X(x)\,dx = \int_0^1 x\left[f_X(x)-f_X(-x)\right]dx = 0.$$

(d) Find $Var(X)$.
Because $E(X) = 0$,

$$Var(X) = E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = 2\int_{-1}^{1} x^2\left(|x| - |x|^3\right)dx$$
$$= 4\int_0^1 \left(x^3 - x^5\right)dx = 4\left(\frac{1}{4} - \frac{1}{6}\right) = \frac{1}{3}.$$

(e) Find $Cov(X, Y)$.
Because $E(X) = E(Y) = 0$ *and* $f_{X,Y}(\cdot,\cdot)$ *is even in each of its arguments,*

$$Cov(X,Y) = E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X,Y}(x,y)\,dydx = \int_{-1}^{1}\int_{-1}^{1} xy f_{X,Y}(x,y)\,dydx$$
$$= \int_0^1\int_0^1 xy\left[f_{X,Y}(x,y) - f_{X,Y}(-x,y) - f_{X,Y}(x,-y) + f_{X,Y}(-x,-y)\right]dydx = 0.$$

(f) Are X and Y independent?
Because the support of $f_{X,Y}(\cdot,\cdot)$ *is the unit disk* $\left\{(x,y) : x^2 + y^2 \leq 1\right\}$, *which is not a rectangle, the random variables* X *and* Y *are not independent. For instance,*

$$0 = P(X \leq x, Y \leq y) \neq P(X \leq x)\,P(Y \leq y) > 0$$

whenever $-1 < x < 0$, $-1 < y < 0$, *and* $x^2 + y^2 > 1$.

2. Prove or disprove (by means of a counterexample) the following: If X and Y are random variables with $Var(X) = Var(Y) = 1$ and $E(Y|X) = X$, then $P(Y = X) = 1$.
By the conditional variance identity and the assumptions,

$$E[Var(Y|X)] = Var(Y) - Var[E(Y|X)] = Var(Y) - Var(X) = 1 - 1 = 0,$$

implying that $Var(Y|X) = 0$. *As a consequence,* $P[Y = E(Y|X) = X] = 1$.

3. Suppose $(X, D)'$ is a bivariate random with $P(D = 0) = P(D = 1) = \frac{1}{2}$, $E(X|D = d) = \mu$ (for $d \in \{0, 1\}$), and

$$Var(X|D = d) = \begin{cases} 1 & \text{if } d = 0, \\ \sigma^2 & \text{if } d = 1, \end{cases}$$

where σ^2 is a known constant and μ is unknown. Define

$$m(X, D; \alpha_0, \alpha_1) = \begin{cases} \alpha_0 X & \text{if } D = 0, \\ \alpha_1 X & \text{otherwise,} \end{cases}$$

where α_0 and α_1 are some constants.

(a) Under what conditions (on α_0 and α_1) is $E[m(X,D;\alpha_0,\alpha_1)] = \mu$ (for every $\mu \in \mathbb{R}$)?
We have:

$$m(X,D;\alpha_0,\alpha_1) \sim \alpha_0 X \cdot 1(D=0) + \alpha_1 X \cdot 1(D=1).$$

Using the law of iterated expectations,

$$\begin{aligned} E[m(X,D;\alpha_0,\alpha_1)] &= E(E[m(X,D;\alpha_0,\alpha_1)|D]) \\ &= E[\alpha_0\mu \cdot 1(D=0) + \alpha_1\mu \cdot 1(D=1)] \\ &= \alpha_0\mu \cdot P(D=0) + \alpha_1\mu \cdot P(D=1) = \frac{1}{2}(\alpha_0+\alpha_1)\mu. \end{aligned}$$

Therefore, $E[m(X,D;\alpha_0,\alpha_1)] = \mu$ *iff* $\alpha_0 + \alpha_1 = 2$.

(b) Assuming $E[m(X,D;\alpha_0,\alpha_1)] = \mu$, find $Var[m(X,D;\alpha_0,\alpha_1)]$.
Using the law of iterated expectations, we have:

$$\begin{aligned} E\left[m(X,D;\alpha_0,\alpha_1)^2\right] &= E\left[\alpha_0^2 E\left(X^2|D=0\right)\cdot 1(D=0) + \alpha_1^2 E\left(X^2|D=1\right)\cdot 1(D=1)\right] \\ &= \frac{1}{2}\left[\alpha_0^2 E\left(X^2|D=0\right) + \alpha_1^2 E\left(X^2|D=1\right)\right] \\ &= \frac{1}{2}\left[\alpha_0^2\left(1+\mu^2\right) + \alpha_1^2\left(\sigma^2+\mu^2\right)\right]. \end{aligned}$$

If $E[m(X,D;\alpha_0,\alpha_1)] = \mu$, *then*

$$\begin{aligned} Var[m(X,D;\alpha_0,\alpha_1)] &= E\left[m(X,D;\alpha_0,\alpha_1)^2\right] - E[m(X,D;\alpha_0,\alpha_1)]^2 \\ &= \frac{1}{2}\left[\alpha_0^2\left(1+\mu^2\right) + \alpha_1^2\left(\sigma^2+\mu^2\right)\right] - \mu^2 \\ &= \frac{1}{2}\left[\alpha_0^2 + \alpha_1^2\sigma^2 + \left(\alpha_0^2+\alpha_1^2-2\right)\mu^2\right]. \end{aligned}$$

(c) Find the values of α_0 and α_1 that minimize $Var[m(X,D;\alpha_0,\alpha_1)]$ subject to the restriction derived in (a).
If $E[m(X,D;\alpha_0,\alpha_1)] = \mu$, *then*

$$\begin{aligned} Var[m(X,D;\alpha_0,\alpha_1)] &= Var[m(X,D;\alpha_0,2-\alpha_0)] \\ &= \frac{1}{2}\left[\alpha_0^2 + (2-\alpha_0)^2\sigma^2 + \left(\alpha_0^2 + (2-\alpha_0)^2 - 2\right)\mu^2\right]. \end{aligned}$$

This minimizing value of α_0 *satisfies*

$$\frac{\partial}{\partial \alpha_0} Var\left[m\left(X,D;\alpha_0,2-\alpha_0\right)\right] = \alpha_0 - \left(2-\alpha_0\right)\sigma^2 + \left(\alpha_0 - \left(2-\alpha_0\right)\right)\mu^2 = 0,$$

which holds when

$$\alpha_0 = \frac{2\left(\sigma^2+\mu^2\right)}{1+\sigma^2+2\mu^2}.$$

The associated value of α_1 *is*

$$\alpha_1 = \frac{2\left(1+\mu^2\right)}{1+\sigma^2+2\mu^2}.$$

(d) Under what conditions (on α_0 and α_1) is $E\left[m\left(X,D;\alpha_0,\alpha_1\right)|D=d\right]=\mu$ for $d\in\{0,1\}$?
From part (a), we have:

$$E\left[m\left(X,D;\alpha_0,\alpha_1\right)|D=d\right] = \alpha_d\mu, \qquad d\in\{0,1\}.$$

Therefore, $E\left[m\left(X,D;\alpha_0,\alpha_1\right)|D=d\right]=\mu$ *for* $d\in\{0,1\}$ *iff* $\alpha_0=\alpha_1=1$.

(e) Under what conditions (on σ^2) does the answer derived in (c) satisfy the condition derived in (d)?
The answer derived in (c) agrees with the answer in (d) iff $\sigma^2=1$.

4. Suppose $X\sim Ber(p)$ for some $p\in(0,1)$; that is, suppose X is a discrete random variable with pmf

$$f(x|p) = p^x(1-p)^{1-x}\,1\left(x\in\{0,1\}\right),$$

where $1(\cdot)$ is the indicator function. Show that $E\left[\log f(X|\theta)\right]\le E\left[\log f(X|p)\right]$ for every $\theta\in(0,1)$. (Hint: Use Jensen's inequality and the fact that $\log(a\cdot b)=\log(a)+\log(b)$.)
By Jensen's inequality (and the fact that $\log(a\cdot b)=\log(a)+\log(b)$*),*

$$\begin{aligned} E\left[\log f(X|\theta)\right] &= E\left(\log\left[\frac{f(X|\theta)}{f(X|p)}\cdot f(X|p)\right]\right) = E\left[\log\frac{f(X|\theta)}{f(X|p)}\right] + E\left[\log f(X|p)\right] \\ &\le \log\left(E\left[\frac{f(X|\theta)}{f(X|p)}\right]\right) + E\left[\log f(X|p)\right]. \end{aligned}$$

Now, because the support of $f(\cdot|p)$ *does not depend on* $p\in(0,1)$,

$$E\left[\frac{f(X|\theta)}{f(X|p)}\right] = \sum_{x\in\mathbb{R}}\frac{f(x|\theta)}{f(x|p)}f(x|p) = \sum_{x\in\mathbb{R}} f(x|\theta) = 1, \qquad \theta\in(0,1).$$

As a consequence,

$$E\left[\log f(X|\theta)\right] \le \log\left(E\left[\frac{f(X|\theta)}{f(X|p)}\right]\right) + E\left[\log f(X|p)\right] = E\left[\log f(X|p)\right].$$