14.381 Waiver Examination

Instructions: This is a closed book exam. You have 90 minutes for the exam. Answer as many questions as possible. Partial answers get partial credit. Please write legibly. *Good luck!*

Let $X_1, \ldots, X_n$ be a random sample from a continuous (uniform) distribution with pdf

$$f(x|\theta) = \frac{1}{\theta} 1(0 \leq x \leq \theta),$$

where $\theta \in \Theta = (0, \infty)$ is an unknown parameter and $1(\cdot)$ is the indicator function.

(a) Show that $\hat{\theta}_{MM} = 2\bar{X}$ is a method of moments estimator of θ (where $\bar{X} = n^{-1} \sum_{i=1}^n X_i$).

(b) Is $\hat{\theta}_{MM}$ unbiased? If "yes", is $\hat{\theta}_{MM}$ a minimum variance unbiased estimator? If "no", find a function $g_{MM}(\cdot)$, not depending on θ, such that $g_{MM}\left(\hat{\theta}_{MM}\right)$ is an unbiased estimator.

(c) Show that $\hat{\theta}_{MM}$ is a consistent estimator.

(d) Find the limiting distribution of $\sqrt{n}\left(\hat{\theta}_{MM} - \theta\right)$ (as $n \to \infty$).

(e) Find the likelihood function and find the maximum likelihood estimator $\hat{\theta}_{ML}$ of θ.

(f) Find the pdf of $\hat{\theta}_{ML}$.

(g) Is $\hat{\theta}_{ML}$ unbiased? If "yes", is $\hat{\theta}_{ML}$ a minimum variance unbiased estimator? If "no", find a function $g_{ML}(\cdot)$, not depending on θ, such that $g_{ML}\left(\hat{\theta}_{ML}\right)$ is an unbiased estimator.

(h) Show that $\hat{\theta}_{ML}$ is a consistent estimator.

(i) Let $PMC\left(\hat{\theta}_{MM}, \hat{\theta}_{ML}\right) = P\left(\left|\hat{\theta}_{MM} - \theta\right| \leq \left|\hat{\theta}_{ML} - \theta\right|\right)$ denote the Pitman measure of closeness of $\hat{\theta}_{MM}$ relative to $\hat{\theta}_{ML}$. Find the limit of $PMC\left(\hat{\theta}_{MM}, \hat{\theta}_{ML}\right)$ (as $n \to \infty$).
[*Hint: Find the limiting distribution of* $\sqrt{n}\left(\hat{\theta}_{ML} - \theta\right)$ *and use the result from (d).*]

(j) Consider the problem of testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$ (where θ_0 is some constant). Show that the test which rejects for large values of $\hat{\theta}_{ML}$ is the uniformly most powerful test (within the class of tests of the same level).