14.381 Final Examination

Fall, 2000

Instructions: This is a closed book exam. You have 180 minutes to answer the questions. Good luck!

1. Let X be a $N(0,\sigma^2)$ random variable conditional on σ^2 with conditional density function
$$f_{X|\sigma^2}(x|\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp(-\frac{1}{2}x^2/\sigma^2).$$
Let σ^2 have a marginal distribution of a χ_1^2 with density function
$$f_{\sigma^2}(z) = \frac{1}{\sqrt{2}\Gamma(1/2)} z^{-1/2}\exp(-\frac{z}{2}).$$
a) Find $E(X|\sigma^2)$ and $E(X)$.
b) Find $E(X^2|\sigma^2)$ and $E(X^2)$.

(Hint: use known facts about the normal distribution).

2. Let X and Y be random variables with a joint density $f_{X,Y}(x,y)$.
a) Show that $Var(X) \geq Var(E(X|Y))$.
b) Show that $Var(X) \geq E(Var(X|Y))$.
c) Show that $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$.

3. Let $e = (1,0,1,0,...)'$ be an $n \times 1$ vector and $a_n = n/2$ if n is even and $a_n = (n+1)/2$ when n is odd. Define $M = I_n - ee'/a_n$.
a) Show that M is symmetric and idempotent.
b) Find the rank of M. State clearly how you obtain the result.
c) Let X be an $n \times 1$ vector of *iid* standard normal random variables with density $(2\pi)^{-1/2}\exp(-\frac{1}{2}x_i^2)$ for the i-th element of X. Find the distribution of $X'MX$.

4. Let $X_t, t = 1,...,2n$ be a sample drawn from a distribution such that
$$\begin{pmatrix} X_{2i-1} \\ X_{2i} \end{pmatrix} \sim N(0,\Sigma)$$
where
$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$
for $|\rho| \leq 1$ and $i = 1,...,n$. Assume that (X_{2i-1}, X_{2i}) is independent of (X_{2j-1}, X_{2j}) for all $i \neq j$.

a) Find the constant a such that

$$\frac{1}{n}\sum_{t=1}^{2n} X_t^2 = a + o_p(1).$$

Make sure that all the assumptions of WLLN's and SLLN's that you might use are satisfied.

b) Find the limiting distribution of

$$S_n = \frac{1}{\sqrt{n}}\sum_{t=1}^{2n} X_t$$

c) Find an approximation to the distribution of $\exp(S_n)$ where S_n is defined in *b*).

5. Let X be a $n \times k$ matrix of constants, β a $k \times 1$ vector of coefficients and ε a $n \times 1$ vector of normal random variables with joint density

$$f(\varepsilon) = (2\pi)^{-n/2}\exp(-\frac{1}{2}\varepsilon'\varepsilon).$$

Let $y = X\beta + \varepsilon$ where y is the $n \times 1$ vector of dependent observations. Assume that you know y and X (this is the data) and you want to find the maximum likelihood estimator $\hat{\beta}$ of β.

a) Write down the likelihood function for the parameter β (Hint: write ε as a function of β).

b) Find the maximum likelihood (ML) estimator $\hat{\beta}$ of β. Is this estimator unbiased? Does it achieve the Cramer-Rao Lowerbound?

For the following questions assume that ε is a vector of *iid* random variables not necessarily from a normal distribution such that $E\varepsilon_i = 0$ and $E\varepsilon_i^2 = 1$ for the i-th element ε_i and for all $i = 1, ..., n$. Assume that $\lim_{n\to\infty} n^{-1}X'X = M$ where M is a positive definite matrix. Also assume that $n^{-1/2}X'\varepsilon \xrightarrow{d} N(0, M)$.

c) Prove that $\hat{\beta}$ is consistent under these assumptions.

d) Find the limiting distribution of $\sqrt{n}(\hat{\beta} - \beta)$. Carefully state all the arguments used.

e) Find the limiting distribution of the statistic $(\hat{\beta} - \beta)'X'X(\hat{\beta} - \beta)$. Make sure to state clearly all known theorems you are using to prove your result.