14.381: Statistics

Final Examination

Instructions: This is a closed book exam. Answer 5 out of the following 7 questions. Write the answers directly on the exam itself. If you need more space, use an exam booklet. Good luck!

Name ____

1. Let $X_1, ..., X_8$ be a random sample, Let $x_1 = -2, x_2 = x_3 = -1, x_4 = x_5 = x_6 = 0, x_7 = 3, x_8 = 8$

   Let the density of x be given by

$$p(x) = \frac{1}{2}e^{-|x-\mu|}$$

a. Show that $p(x)$ is a density.

b. Estimate μ by maximum likelihood

c. Propose a test (any test) for testing $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ and derive the asymptotic distribution of your test statistic.

d. Give the empirical distribution function.

2. Let $\{X_1, Y_1\}, \{X_2, Y_2\}, ..., \{X_{30}, Y_{30}\}$ be a random sample.

Let $\frac{1}{30}\sum_i X_i = 0 \quad \frac{1}{30}\sum_i Y = 0 \quad \frac{1}{30}\sum_i X_i Y_i = 1 \quad \frac{1}{30}\sum_i X_i^2 = 2$

1. Let the density of Y given X be given by

$$p(y \mid x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma^2} e^{-\frac{1}{2}\frac{(Y-\alpha-x\beta)^2}{\sigma^2}}$$

   Test $H_0 : \alpha = \beta = 0$ against $H_1 : H_0$ not true.

   You may use the following: Let w_1, w_2, w_3, w_4 have chi-square distributions with 1,2,3, and 4 degrees of freedom, then

$$\begin{aligned} p(w_1 &\geq 3.84) = p(w_2 \geq 5.99) = p(w_3 \geq 7.81) \\ &= p(w_4 \geq 9.49) = 0.95 \end{aligned}$$

3. Let $X \sim$Gamma$(\alpha, 1)$

   Let $X_1, ..., X_{100}$ be a random sample and let

$$\frac{\Sigma}{N} \ln X_i = -0.57 \quad \frac{\Sigma}{N} X_i = 1.1 \quad \frac{\Sigma}{N} X_i^2 = 2.21$$

   a. Estimate α using maximum likelihood and test $H_0 : \alpha = 2$ against $H_1 : \alpha \neq 2$. Choose a confidence level that you like.

   b. Estimate α using method of moments and test $H_0 : \alpha = 2$ against $H_1 : \alpha \neq 2$ using your method of moment estimate. Choose a confidence level that you like.

   You may use the following

$$\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1) = \int_0^\infty w^{\alpha-1} e^{-w} dw$$

$$\psi(\alpha) = \tfrac{d\Gamma(\alpha)}{d\alpha} \quad \psi(\alpha) = \tfrac{1}{\alpha-1} + \psi(\alpha - 1)$$
$$\psi'(\alpha) = \tfrac{d\psi(\alpha)}{d\alpha} \quad \psi'(\alpha) = -\tfrac{1}{(\alpha-1)^2} + \psi'(\alpha - 1)$$

$$\Gamma(1) = 1 \quad \psi(1) = -0.57 \quad \psi'(1) = 1.64$$

   Let z have a standard normal distribution with cdf $F(\cdot)$. Then $F(1.96) = 0.975$. Let t have a t-distribution with 99 degrees of freedom with cdf $F(\cdot)$. Then $F(1.96) = 0.975$.

4. Let $T^{\alpha_0} = Z$ where $Z \sim$Gamma$(1, 1)$. Show how to estimate α_0 using maximum likelihood and prove consistency of the maximum likelihood estimator.

5. Let $Y \sim$Gamma(α, β). Let $\frac{\Sigma}{N} \ln Y_i = -1$ and $\frac{\Sigma}{N} Y_i = 2$. Test $H_0 : \alpha = \beta = 1$. You may use the information at the end of question 2.

6. Let $E\varepsilon = 0$ and $E\varepsilon^2 = \sigma^2$. Let $\varepsilon_1, ..., \varepsilon_N$ be a raondom sample. Consider the following proposition:

   Proposition 1: $\frac{\sum_i \varepsilon_i}{N^{3/2}\sigma} \underset{d}{\rightarrow} N(0, v)$ where: $v > 0$.

   Prove the proposition and determine v.

7. Let $Y_i = X_i\gamma + \varepsilon_i$ where $X_i = \left\{ \begin{matrix} 1 \\ i \end{matrix} \right\}$; and $\varepsilon_1, ...\varepsilon_N$ are randomly drawn from a distribution with mean zero and variance σ^2.

   a. Show that the least squares estimator is a consistent estimator for γ.

   b. Derive the rate of convergence and the asymptotic distribution of the estimator of the slope parameter (i.e. the second element of γ). Feel free to express the asymptotic distribution in terms of v.