14.381 STATISTICS
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HANDOUT #10 + CORRECTION

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1. CdB 8.28

$$f(x|\theta) = \frac{e^{(x-\theta)}}{(1+e^{(x-\theta)})^2}, \quad -\infty < x < \infty, \quad -\infty < \theta < \infty$$

logistic location pdf.

(a) Show that this family has a MLR
For $\theta_2 > \theta_1$, the likelihood ratio is

$$\frac{f(x|\theta_2)}{f(x|\theta_1)} = e^{\theta_1 - \theta_2}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right]^2$$

$$\frac{d}{dx}\left[\frac{1+e^{x-\theta_1}}{1+e^{x-\theta_2}}\right] = \frac{e^{x-\theta_1} - e^{x-\theta_2}}{(1+e^{x-\theta_2})^2}.$$

Because $\theta_2 > \theta_1$, $e^{x-\theta_1} > e^{x-\theta_2}$, and, hence, the ratio is increasing. This family has MLR.

(b) The best test is to reject H_0 if $f(x|1)/f(x|0) > k$. From part (a), this ratio is increasing in x. Thus this inequality is equivalent to rejecting if $x > k'$. The cdf of this logistic is

$$F(x|\theta) = \frac{e^{x-\theta}}{(1+e^{x-\theta})}$$

Thus,

$$\alpha = 1 - F(k'|0) = \frac{1}{1+e^{k'}}$$

and

$$1-\beta = F(k'|1) = \frac{e^{k'-1}}{1+e^{k'-1}}$$

For a specified α, $k' = \log[(1-\alpha)/\alpha]$.

So for $\alpha = 0.2$, $k' \approx 1.386$ and $\beta \approx .595$

(c) The Karlin-Rubin Theorem is satisfied, so the test is the UMP of its size.
We can say that we can apply Karlin-Rubin!

② C&B 9.2
Let $X_1, \ldots, X_n$ be iid $N(\theta, 1)$. A 95% confidence interval for θ is $\bar{x} \pm 1.96/\sqrt{n}$. Let p denote the probability that an additional independent observation, X_{n+1}, will fall in this interval. Is p greater than, less than, or equal to .95? Prove your answer.

③ C&B 9.3
The independent random variables $X_1, \ldots, X_n$ have the common distribution

$$P(X_i \le x) = \begin{cases} 0 & \text{if } x \le 0 \\ (x/\beta)^\alpha & \text{if } 0 < x < \beta \\ 1 & \text{if } x \ge \beta \end{cases}$$

(a) Assume $\alpha = \alpha_0$ is known. Find the MLE of β.
(b) Find an upper confidence limit for β with confidence coefficient .95.

See solution in the last page

(a) The MLE of β is $X_{(n)} = \max\{X_i\}$.

(b) Proceed in the following way

$$.05 = P_\beta(X_{(n)}/\beta \le c) = P_\beta(\text{all } X_i \le \beta c)$$

$$= \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n} = c^{\alpha_0 n}$$

$$\Rightarrow \quad c = .05^{1/\alpha_0 n}.$$

Thus $.95 = P_\beta(X_{(n)}/\beta \le c) = P_\beta(X_{(n)}/c > \beta)$, and $\{\beta : \beta < X_{(n)}/(.05^{1/\alpha_0 n})\}$ is a 95% upper confidence limit for β

X

(4) C&B 9.16

(a) The LRT has rejection region

$$\{x : |\bar{x} - \theta_0| > z_{\alpha/2}\,\sigma/\sqrt{n}\}$$

acceptance region $A(\theta_0) = \{x : -z_{\alpha/2}\,\sigma/\sqrt{n} \le \bar{x} - \theta_0 \le z_{\alpha/2}\,\sigma/\sqrt{n}\}$,

and $1-\alpha$ confidence interval

$$C(\theta) = \{\theta : \bar{x} - z_{\alpha/2}\,\sigma/\sqrt{n} \le \theta \le \bar{x} + z_{\alpha/2}\,\sigma/\sqrt{n}$$

(b) We have a UMP test with rejection region

$$\{x : \bar{x} - \theta_0 < -z_\alpha\,\sigma/\sqrt{n}\},$$

acceptance region $A(\theta_0) = \{x : \bar{x} - \theta_0 \ge -z_\alpha\,\sigma/\sqrt{n}\}$, and $1-\alpha$ confidence interval

$$C(\theta) = \{\theta : \bar{x} + z_\alpha \sigma/\sqrt{n} \geq \theta\}$$

(c) Similar to (b)

UMP test: rejection region $\{x : \bar{x} - \theta_0 > z_\alpha \sigma/\sqrt{n}\}$
acceptance region $\{x : \bar{x} - \theta_0 \leq z_\alpha \sigma/\sqrt{n}\} = A(\theta_0)$
$1-\alpha$ confidence interval $\{\theta : \bar{x} - z_\alpha \sigma/\sqrt{n} \leq \theta\} = C(\theta)$

⑤ C&B 9.17
Find a $1-\alpha$ confidence interval for θ, given $x_1, \dots, x_n$ iid with pdf

(a) $f(x|\theta) = 1, \quad \theta - \frac{1}{2} < x < \theta + \frac{1}{2}$

(b) $f(x|\theta) = \dfrac{2x}{\theta^2}, \quad 0 < x < \theta, \quad \theta > 0$

C&B 9.13

CORRECTION:

$$P(X_i \leq x) = \begin{cases} 0 & \text{if } x \leq 0 \\ (x/\beta)^\alpha & \text{if } 0 < x < \beta \\ 1 & \text{if } x \geq \beta \end{cases}$$

$$f(x) = \frac{\alpha x^{\alpha-1}}{\beta^\alpha} \cdot \mathbb{1}(x \in (0,\beta)) \qquad \Rightarrow \quad \alpha > 0$$

$$L(X_1, \ldots, X_n \mid \beta) = \frac{\alpha^n}{\beta^{\alpha n}} \prod_{i=1}^{n} \left[X_i^{\alpha-1} \mathbb{1}(X_i \in (0,\beta)) \right]$$

$$= \left(\frac{\alpha}{\beta^\alpha}\right)^n \prod_{i=1}^{n} X_i^{\alpha-1} \cdot \mathbb{1}(X_{(n)} \in (0,\beta))$$

$$= \left(\frac{\alpha}{\beta^\alpha}\right) \underbrace{\prod_{i=1}^{n} X_i^{\alpha-1}}_{\substack{\text{does not} \\ \text{depend on } \beta}} \cdot \mathbb{1}(X_{(n)} \leq \beta)$$

To find the MLE we just look at

$$\left(\frac{\alpha}{\beta^\alpha}\right)^n \mathbb{1}(X_{(n)} \leq \beta)$$

As $\alpha > 0$, $(\alpha/\beta^\alpha)^n$ is a decreasing function of $\beta \Rightarrow$ you want to reduce β as much as possible. You can do it up to the point where $\beta = \max\{X_i\}$

$$\hat{\beta}_{ML} = \max\{X_i\} = X_{(n)}$$

(b) $0.05 = P_\beta(X_{(n)}/\beta \le c) = P_\beta(\text{all } X_i \le c\beta)$

$$= \left(\frac{c\beta}{\beta}\right)^{\alpha_0 n}$$

$$= c^{\alpha_0 n}$$

So $c = 0.05^{1/\alpha_0 \cdot n}$.

Thus, $0.95 = P_\beta(X_{(n)}/\beta > c) = P_\beta(X_{(n)}/c > \beta)$, and $\{\beta : \beta < X_{(n)}/(0.05^{1/\alpha_0 n})\}$ is a 95% upper confidence limit for β.

Now, you may still wonde why I started using

$$0.05 = P_\beta(X_{(n)}/\beta \le c).$$

Think in the following way: for a given $X_{(n)}$, the larger the β, the smaller the ratio $X_{(n)}/\beta$. So we are implicitly looking for a β sufficiently large ... "sufficiently large" meaning, making $X_{(n)}$ very implausible.

You can also think about inverting a LRT, and you would get a very similar intuition.