14.381 Midterm Examination
Fall, 1999

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Make sure to show all derivations needed for your results. Good luck!

1. Show that it is not possible to find events A, B, C such that
$$P(A) = P(B) = P(C) = \frac{3}{8}$$
and
$$P(A \cup B) = P(A \cup C) = P(B \cup C) = \frac{3}{4}.$$

2. Show that for two random variables X and Y such that $E\left|X\right|^2 < \infty$, $E\left|Y\right|^2 < \infty$ it follows that
$$\operatorname{var}(E(X \mid Y)) \leq \operatorname{var}(X).$$

3. Let X be a normal random variable with density $f_X(x) = (2\pi)^{-1/2}\sigma^{-1}e^{-1/2(x-\mu)/\sigma^2}$ where $\mu = EX$ and $\sigma^2 = \operatorname{var}(X)$. Let K_i be the i-th cumulant of X. Let $Y = \exp(X)$.

   Find

   a) EY,
   b) EY^2,
   c) EY^r, $r > 0$,
   d) K_1,
   e) K_2,
   f) K_i $i \geq 3$, i integer

4. Let $X_1, ..., X_N$ be a collection of mutually independent random variables with $X_i \sim N(0, \sigma_i)$ where σ_i is a $\chi^2_{(2)}$ random variable. Define $S_N = \sum_{i=1}^N X_i$ with N a constant integer $N > 0$. Find $E(S_N)$ and $\operatorname{var}(S_N)$ (Hint: you can use the fact that the moment generating function for the $\chi^2_{(2)}$ distribution is $1/(1-2t)$ to obtain the moments for σ_i).

5. Let X_1 and X_2 be independent random variables with densities
$$f_{X_i}(x) = \begin{cases} \frac{\lambda^{n_i} x^{n_i - 1}}{\Gamma(n_i)} e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n_i > 0, \lambda > 0, i = 1, 2$$

   Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1 / X_2$.

   a) Find the joint density of Y_1 and Y_2.
   b) Show that Y_1 and Y_2 are independent.