14.381 Midterm Examination
Fall, 1999

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Make sure to show all derivations needed for your results. Good luck!

1. Show that it is not possible to find events $A$, $B$, $C$ such that

$$P(A) = P(B) = P(C) = \frac{3}{8}$$

and

$$P(A \cup B) = P(A \cup C) = P(B \cup C) = \frac{3}{4}.$$ 

2. Show that for two random variables $X$ and $Y$ such that $E |X|^2 < \infty, E |Y|^2 < \infty$ it follows that \( \text{var}(E(X | Y)) \leq \text{var}(X). \)

3. Let $X$ be a normal random variable with density $f_X(x) = (2\pi)^{-1/2} \sigma^{-1} e^{-1/2(x-\mu)/\sigma^2}$ where $\mu = EX$ and $\sigma^2 = \text{var}(X)$. Let $K_i$ be the i-th cumulant of $X$. Let $Y = \exp(X)$.

Find

a) $EY$, 
b) $EY^2$, 
c) $EY^r$, $r > 0$, 
d) $K_1$, 
e) $K_2$, 
f) $K_i$, $i \geq 3$, $i$ integer

4. Let $X_1, ..., X_N$ be a collection of mutually independent random variables with $X_i \sim N(0, \sigma_i)$ where $\sigma_i$ is a $\chi^2_{(2)}$ random variable. Define $S_N = \sum_{i=1}^N X_i$ with $N$ a constant integer $N > 0$. Find $E(S_N)$ and $\text{var}(S_N)$ (Hint: you can use the fact that the moment generating function for the $\chi^2_{(2)}$ distribution is $1/(1 - 2t)$ to obtain the moments for $\sigma_i$).

5. Let $X_1$ and $X_2$ be independent random variables with densities

$$f_{X_i}(x) = \begin{cases} \frac{\lambda_i x^{n_i-1}}{(n_i)!} e^{-\lambda x} & x \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n_i > 0, \lambda > 0, i = 1, 2$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

a) Find the joint density of $Y_1$ and $Y_2$.

b) Show that $Y_1$ and $Y_2$ are independent.