14.381 STATISTICS

FALL 2004

PRACTICE PROBLEMS

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Statistics 14.381

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Fall 2002

Midterm Examination

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Carefully derive or explain all your solutions. You can use known facts about distributions established in class without proof, but make sure to explain why such facts apply in your case. Good luck!

1. a) Let A and B be events with positive probability. Show that $P(A^c|B) = 1 - P(A|B)$.

   b) Let $A_1, ..., A_n$ be events. Show that
$$P\left(\cap_{i=1}^n A_i\right) \geq \sum_{i=1}^n P(A_i) - (n-1).$$

2. Let X and Y be random variables such that $EX^2 < \infty$ and $EY^2 < \infty$. Assume that there exist functions $h(Y)$ and $g(Y)$ such that $E(h(Y)^2) < \infty$ and $E(g(Y)^2) < \infty$. Define $f(Y,a) = ag(Y) + h(Y)$ where a is some constant. Assume that $E(g(Y)(X - h(Y)) \neq 0$.

   a) Find the constant a such that
$$a = \arg\min_a E(X - f(Y,a))^2.$$

   b) Show that $h(Y)$ can not be the conditional expectation of X, i.e. show that $h(Y) \neq E(X|Y)$ by showing that there exists some function $\tilde{h}(Y)$ such that $E(X - \tilde{h}(Y))^2 < E(X - h(Y))^2$.

3. Let $X = [X_1, X_2]'$ have a bivariate normal distribution with parameters $\mu = 0$ and
$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$
   The density of X is $f_X(x) = (2\pi)^{-1}(\det\Sigma)^{-1/2}\exp(-1/2x'\Sigma^{-1}x)$. Find expressions for

   a) $Ee^{t'X}$ where $t' = [t_1, t_2]$ and t_1 and t_2 are constants.

   b) EX_1X_2.

   c) EX_1^3.

   d) $EX_1X_2^2$.

4. Let X be a $n \times 1$ vector of random variables with $EX = \mu$ and $E(X - \mu)(X - \mu)' = \Sigma$. Note that X is not assumed to be multivariate normal. Let a and b be $n \times 1$ vectors and c a constant scalar. Assume that $a'b = 0$. Let $Y = a'X + c$ and $Z = b'X$

   a) Find the mean and variance of $Y = a'X + c$.

   b) Find $\text{cov}(Y,Z)$. Are Y and Z independent?

5. The joint density of X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} \frac{e^{-\frac{x}{2}} x^{\frac{r}{2}-2}}{(2)^{\frac{r}{2}}\Gamma(\frac{r}{2})} & \text{if } 0 < y < x < \infty \\ 0 & \text{elsewhere} \end{cases}$$

for constants $r = 1, 2, ...$ and $\Gamma(r)$ the Gamma function.

a) Find the marginal density of X. Indicate the range on which the marginal density is defined.

b) Find the conditional density of Y given $X = x$. Make sure to indicate the range on which the conditional density is defined.

c) Find EX and EY.

MIT, Department of Economics
Statistics 14.381

Guido Kuersteiner
Fall 2001

Midterm Examination

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Good luck!

1. Let A and B be events with positive probability. Show the following:
   a) If A and B are independent then A and B^c are independent.
   b) If $P(A|B) < P(A)$ then $P(B|A) < P(B)$.

2. Prove or disprove the following: If $E(Y|X) = X$ and $E(X|Y) = Y$ and both EX^2 and EY^2 are finite then $P[X = Y] = 1$ (Hint: Show that $\text{Var}(X - Y) = 0$).

3. Let $X = [X_1, X_2]'$ have a bivariate normal distribution with parameters $\mu = 0$ and
$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}.$$
   a) Write down the marginal density of X_1 and the conditional density of X_1 given X_2. Find the first two (conditional) moments of the conditional distribution (Hint: you can use known properties of the normal distribution without proof).
   b) Let $\varepsilon(b) = X_1 - bX_2$. Find the value for the parameter b, expressed in terms of the parameters Σ, such that $E[\varepsilon(b)]^2$ is minimal.

4. Let X be a $n \times 1$ vector of random variables with $EX = \mu$ and $E(X - \mu)(X - \mu)' = \Sigma$. Note that X is not assumed to be multivariate normal.
   a) Let A be a $n \times n$ matrix and b an $n \times 1$ vector. Find the mean vector and covariance matrix of $Y = AX + b$.
   b) Show that $EX'AX = \text{tr}(A(\Sigma + \mu\mu'))$.

5. Let X and Y be random variables with joint density
$$f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}.$$
   Let $Z = X + Y$.
   a) Find the joint density of Z and X (Hint: be careful to state where the density is defined).
   b) Find the marginal density of Z and the marginal density of X.
   c) Find the conditional density of Z given $X = x$.

1. (a) $P(A \cap B^c) = P(A \cap \Omega \setminus B) = P(A \cap \Omega) - P(A \cap B)$

$= P(A) - \underbrace{P(A)P(B)}_{A,B \text{ indep.}} = P(A)[1 - P(B)]$

$= P(A)P(B^c)$

(b) $P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$

$\Rightarrow \dfrac{P(B)}{P(B|A)} = \dfrac{P(A)}{P(A|B)} > 1 \Rightarrow P(B) > P(B|A)$

2. $Var(X-Y) = E(Var(X-Y|X)) + Var(E(X-Y|X))$

$= E(Var(X-X)) + Var(E(X-X))$

$= E(Var(0)) + Var(E(0)) = 0 + 0 = 0.$

$E(X-Y) = E(E(X-Y|X)) = E(E(X-X)) = E(0) = 0$

$\Rightarrow$ X, Y have same expectation, their difference has 0 varience $\Rightarrow$

$P(X=Y) = 1.$

3. (a) $X_1 \sim N(\mu_1, Var(X_1)) = N(0, \sigma_1^2)$

$$f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma_1} e^{-\frac{1}{2}(x_1/\sigma_1)^2}.$$

$$X_1|X_2 \sim N(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(X_2-\mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21})$$

$$= N(\rho\sigma_1/\sigma_2 (X_2), \sigma_1^2 - \rho^2\sigma_1^2)$$

$$= N\left(\frac{\rho\sigma_1}{\sigma_2} X_2, (1-\rho^2)\sigma_1^2\right)$$

$$f_{X_1|X_2}(x_1|x_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \cdot \frac{1}{\sigma_1} e^{-\frac{1}{2}\frac{(X_1 - \frac{\rho\sigma_1}{\sigma_2}X_2)^2}{(1-\rho^2)\sigma_1^2}}$$

$$E(X_1|X_2) = \frac{\rho\sigma_1}{\sigma_2} X_2$$

$$E(X_1^2|X_2) = Var(X_1|X_2) + E(X_1|X_2)^2$$

$$= (1-\rho^2)\sigma_1^2 + \left(\frac{\rho\sigma_1}{\sigma_2} X_2\right)^2$$

$$= \sigma_1^2 - \rho^2\sigma_1^2 + \rho^2\sigma_1^2/\sigma_2^2 \, X_2^2$$

$$= \boxed{\sigma_1^2\left[1 + \rho^2\left(X_2^2/\sigma_2^2 - 1\right)\right]}$$

(b) $\underset{b}{\operatorname{argmin}}\ E(X_1 - bX_2)^2 = E(X_1^2 - 2bX_1X_2 + b^2X_2^2)$

$$\Rightarrow \underset{b}{\operatorname{argmin}}\ E(X_1^2) - 2b\,E(X_1X_2) + b^2 E(X_2^2)$$

FOC: $-2E(X_1X_2) + 2b\,E(X_2^2) = 0$

$$\Rightarrow b = \frac{E(X_1X_2)}{E(X_2^2)} = \frac{Cov(X_1, X_2)}{Var(X_2)} = \frac{\rho\sigma_1\sigma_2}{\sigma_2^2} = \boxed{\frac{\rho\sigma_1}{\sigma_2}}$$

4. (a) $Y = AX + b$

$E(Y) = A E(X) + b = A\mu + b$

$Var(Y) = A\, Var(X)\, A' = A \Sigma A'$

(b)
- $\underset{(1\times n)(n\times n)(n\times 1)}{X'AX}$ is 1×1 so $X'AX = tr(X'AX)$
- By the property of the trace, $tr(X'AX) = tr(AXX')$ (cyclical property of square matrices)
- $\Sigma = E(X-\mu)(X-\mu') = E(XX' - \mu X' - X\mu' + \mu\mu')$
  $= E(XX') - \mu\mu' - \mu\mu' + \mu\mu' = E(XX') - \mu\mu'$
  $\Rightarrow E(XX') = \Sigma + \mu\mu'$
- Finally,

$$
\begin{aligned}
E(X'AX) &= E(tr(X'AX)) \\
&= E(tr(AXX')) \\
&= tr(E(AXX')) \\
&= tr(A E(XX')) \\
&= tr(A(\Sigma + \mu\mu')).
\end{aligned}
$$

5. $f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & x>0,\ y>0 \\ 0 & \text{otherwise} \end{cases}$

$Z = X + Y$

(a) $Z = X + Y \qquad V = X$

$\Rightarrow X = V \qquad Y = Z - V, \qquad X > 0 \Rightarrow V > 0$

$Y > 0 \Rightarrow Z - V > 0 \Rightarrow Z > V$

$$J = \begin{vmatrix} \frac{dX}{dV} & \frac{dX}{dZ} \\ \frac{dY}{dV} & \frac{dY}{dZ} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$$

$$f_{Z,V} = f_{X,Y}(v, z-v) = e^{-(v+z-v)} = e^{-z}$$

$$\Rightarrow f_{Z,X} = \begin{cases} e^{-z} & 0 < x < z \\ 0 & \text{otherwise} \end{cases}$$

b) $f_Z = \int_0^z e^{-z}\, d\cancel{x} = x e^{-z} \Big|_0^z = z e^{-z}, \quad z > 0$

$f_X = \int_x^\infty e^{-z}\, dz = -e^{-z} \Big|_x^\infty = 0 + e^{-x}$

$= e^{-x}, \quad x > 0$

c) $f_{Z|X} = f_{X,Z} / f_X = e^{-z} / e^{-x} = e^{x-z}$

$$= \begin{cases} e^{-(z-x)} & 0 < x < z \\ 0 & \text{otherwise} \end{cases}$$

14.381 Midterm Examination
Fall, 1999

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Make sure to show all derivations needed for your results. Good luck!

1. Show that it is not possible to find events A, B, C such that
$$P(A) = P(B) = P(C) = \frac{3}{8}$$
and
$$P(A \cup B) = P(A \cup C) = P(B \cup C) = \frac{3}{4}.$$

2. Show that for two random variables X and Y such that $E|X|^2 < \infty$, $E|Y|^2 < \infty$ it follows that
$$\text{var}(E(X \mid Y)) \leq \text{var}(X).$$

3. Let X be a normal random variable with density $f_X(x) = (2\pi)^{-1/2}\sigma^{-1}e^{-1/2(x-\mu)/\sigma^2}$ where $\mu = EX$ and $\sigma^2 = \text{var}(X)$. Let K_i be the i-th cumulant of X. Let $Y = \exp(X)$.

   Find

   a) EY,
   b) EY^2,
   c) EY^r, $r > 0$,
   d) K_1,
   e) K_2,
   f) K_i $i \geq 3$, i integer

4. Let $X_1, ..., X_N$ be a collection of mutually independent random variables with $X_i \sim N(0, \sigma_i)$ where σ_i is a $\chi^2_{(2)}$ random variable. Define $S_N = \sum_{i=1}^N X_i$ with N a constant integer $N > 0$. Find $E(S_N)$ and $\text{var}(S_N)$ (Hint: you can use the fact that the moment generating function for the $\chi^2_{(2)}$ distribution is $1/(1-2t)$ to obtain the moments for σ_i).

5. Let X_1 and X_2 be independent random variables with densities
$$f_{X_i}(x) = \begin{cases} \frac{\lambda^{n_i} x^{n_i - 1}}{\Gamma(n_i)} e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n_i > 0, \lambda > 0, i = 1, 2$$

   Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

   a) Find the joint density of Y_1 and Y_2.
   b) Show that Y_1 and Y_2 are independent.

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14.381 Problem Set 6
Fall, 2002

Practice Problems - No Due Date

1. (Midterm 1999) Let X be a normal random variable with density

$$f_X(x) = (2\pi)^{-1/2}\sigma^{-1}e^{-1/2(x-\mu)/\sigma^2}$$

Define $\mu = EX$, $\sigma^2 = Var(X)$, $Y = \exp(X)$. Let K_i be the i-th cumulant of X. Evaluate:

   (a) $E(Y)$
   (b) $E(Y^2)$
   (c) $E(Y^r)$, $r > 0$
   (d) K_1
   (e) K_2
   (f) K_i; $i \geq 3$, i integer

2. (Waiver 1999) Given N, let $X_1, ..., X_N$ be an *iid* sample of normal random variables with mean μ and variance σ^2 . Let N, the sample size, be a Poisson random variable

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!} \; for\ n = 0, 1, 2, ..., \; \lambda > 0$$

Consider the sum of the observations

$$S = \left\{ \begin{array}{ll} \sum_{i=1}^{N} X_i & N > 0 \\ 0 & N = 0 \end{array} \right\}$$

Find ES and ES^2

(Hint: you can use the fact that $EN = \lambda$ and $Var(N) = \lambda$ without proof).

3. (Midterm 1998) We observe a process of events occuring randomly in time. Let N_t be the total number of events between 0 and t. N_t is distributed as a poison random variable with parameter λ, i.e.

$$P(N_t = k) = \frac{e^{-\lambda t}(\lambda t)^k}{k!}$$

Also assume that N_t and $N_{t+s} - N_t$ are independent for all t, $s > 0$ and that $N_{t+s} - N_t$ is poisson with $P(N_{t+s} - N_t = k) = \frac{e^{-\lambda s}(\lambda s)^k}{k!}$

(a) Find EN_t for some t fixed.

(b) Find $E(N_{t+s}|N_t)$ for $t, s > 0$ fixed

4. (Midterm 1998) Assume that X is a normal r. v. with moment generating function $e^{t\mu+\frac{1}{2}t^2\sigma^2}$. Find the following moments:

   (a) $E(X^2)$

   (b) $E(X^3)$

   (c) $E((X-\mu)^r)$ for $r > 0$; r integer

   (d) $E(e^X)$

5. (Midterm 1998) If the joint density of (X, Y) is

$$f_{X,Y}(x,y) = \left\{ \begin{array}{ll} \lambda e^{-\lambda y}\sqrt{\frac{\lambda}{\pi}} & x^2 < y < \infty, \; -\infty < x < \infty \\ 0 & elsewhere \end{array} \right\}$$

   Find:

   (a) The marginal density of X

   (b) The conditional density of Y given $X = x$

6. Two pennies, one with $P(head) = u$ and one with $P(head) = w$, are to be tossed together independently. Define $p_0 = P(0$ heads occur$)$, $p_1 = P(1$ head occurs$)$, $p_2 = P(2$ heads occur$)$. Can u and w be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

7. Betteley (1977) provides an interesting addition law of expectations. Let X and Y be any two random variables and define $X \wedge Y = \min(X, Y)$ and $X \vee Y = \max(X, Y)$. Analogous to the probability law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, show that $E(X \vee Y) = EX + EY - E(X \wedge Y)$.

   (Hint: Establish that $X + Y = (X \wedge Y) + (X \vee Y)$)

8. For any three random variables X, Y, and Z with finite variances, prove the covariance identity $Cov(X,Y) = E(Cov(X,Y|Z)) + Cov(E(X|Z), E(Y|Z))$.

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Problem Set 6

(Practice Problems)

Solutions

1. $f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2}$, $Y = e^X$

(a) $E(Y) = E(e^X) = M_X(1) = e^{\mu t + 1/2 t^2 \sigma^2}\Big|_{t=1}$

$= e^{\mu + 1/2\sigma^2}$

(b) $E(Y^2) = E(e^{2X}) = M_X(2) = e^{2\mu + 2\sigma^2}$

(c) $E(Y^r) = E(e^{rX}) = M_X(r) = e^{r\mu + 1/2 r^2\sigma^2}$

(d) $K_1 = \frac{\partial K_X(t)}{\partial t}\Big|_{t=0}$

$K_X(t) = \log(M_X(t)) = \log\left(e^{t\mu + 1/2 t^2\sigma^2}\right)$

$= \mu t + 1/2\, t^2\sigma^2$

$K_1 = (\mu + t\sigma^2)\big|_{t=0} = \mu$

(e) $K_2 = \frac{\partial^2 K_X(t)}{\partial t^2}\Big|_{t=0} = \sigma^2\big|_{t=0} = \sigma^2$

(f) K_i, $i \geq 3$, i integer

$K_i = \frac{\partial^i K_X(t)}{\partial t^i}\Big|_{t=0} = 0 \quad \forall\, i \geq 3$ integer.

2. $X_i \mid N \sim N(\mu, \sigma^2) \quad \forall\, i$

$X_i \mid N$ independent across observations

$P(N=n) = (e^{-\lambda}\lambda^n)/n!$

$$S = \begin{cases} \sum_{i=1}^{N} X_i & N > 0 \\ 0 & N = 0 \end{cases}$$

$$E(S) = E\left(\sum_{i=1}^{N} X_i\right) = E\left(E\left(\sum_{i=1}^{N} X_i \mid N\right)\right)$$

$$= E\left(\sum_{i=1}^{n} E(X_i \mid N)\right) = E(N\mu) = \mu E(N) = \mu\lambda$$

$$E(S^2) = Var(S) + (E(S))^2$$

$$Var(S) = Var\left(\sum_{i=1}^{n} X_i\right) = Var\left(E\left(\sum_{i=1}^{n} X_i \mid N\right)\right) + E\left(Var\left(\sum_{i=1}^{n} X_i \mid N\right)\right)$$

$$= Var(N\mu) + E\left(\sum_{i=1}^{n} Var(X_i \mid N)\right)$$

$$= \mu^2 Var(N) + E(N\sigma^2)$$ because X_i are iid

$$= \mu^2\lambda + \sigma^2\lambda$$

$$E(S^2) = \mu^2\lambda + \sigma^2\lambda + \mu^2\lambda^2 = \lambda(\mu^2(1+\lambda) + \sigma^2)$$

3. $P(N_t = k) = (e^{-\lambda t}(\lambda t)^k)/k!$

N_t, $N_{t+s} - N_t$ independent $\forall$ $t, s > 0$,

$N_{t+s} - N_t$ is poisson : $P(N_{t+s} - N_t = k) = \dfrac{e^{-\lambda s}(\lambda s)^k}{k!}$

(a) $E(N_t) = \sum_{k=0}^{\infty} k e^{-\lambda t}(\lambda t)^k / k!$

$= \sum_{k=1}^{\infty} e^{-\lambda t}(\lambda t)^k / (k-1)! = \lambda t \underbrace{\sum_{k=1}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!}}_{=1} = \lambda t$

(b) $E(N_{t+s} | N_t) = E(N_{t+s} - N_t | N_t) + E(N_t | N_t)$

$= E(N_{t+s} - N_t) + E(N_t | N_t)$

by independence

$= \underbrace{\lambda s}_{\text{using result from (a).}} + N_t$

$E(N_{t+s} | N_t)$ is a random variable - it is a function of N_t.

4. (a) $E(X^2) = M_X^{(2)}(0) = \left.\frac{\partial^2 M_X(t)}{\partial t^2}\right|_{t=0}$

$$= \left.\frac{\partial \left(e^{t\mu + 1/2 t^2 \sigma^2}(\mu + t\sigma^2)\right)}{\partial t}\right|_{t=0}$$

$$= \left. e^{t\mu + 1/2 t^2\sigma^2}\left((\mu + t\sigma^2)^2 + \sigma^2\right)\right|_{t=0}$$

$$= \mu^2 + \sigma^2$$

(b) $E(X^3) = \left.\frac{\partial\, e^{t\mu + 1/2 t^2\sigma^2}\left((\mu + t\sigma^2)^2 + \sigma^2\right)}{\partial t}\right|_{t=0}$

$$= \left. e^{t\mu + 1/2 t^2\sigma^2}\left((\mu + t\sigma^2)^3 + (\mu + t\sigma^2)\sigma^2 + 2(\mu + t\sigma^2)\sigma^2\right)\right|_{t=}$$

$$= e^{t\mu + 1/2 t^2\sigma^2}\left[(\mu + t\sigma^2)^3 + 3(\mu + t\sigma^2)\sigma^2\right]_{t=0}$$

$$= \mu^3 + 3\mu\sigma^2$$

(c) $E(X-\mu)^r$ for $r > 0$, r integer

Define $Y = X - \mu$

$M_Y(t) = E(e^{Yt}) = E(e^{(X-\mu)t}) = e^{-\mu t}E(e^{Xt})$

$$= e^{-\mu t} e^{t\mu + 1/2 t^2\sigma^2} = e^{1/2 t^2\sigma^2}$$

$E(X-\mu)^r = E(Y^r) = \left.\frac{\partial^r e^{1/2 t^2\sigma^2}}{\partial t^r}\right|_{t=0}$

4. (c) We want to evaluate the r^{th} derivative of $e^{1/2 t^2 \sigma^2}$ at $t=0$.

$$e^{1/2 t^2\sigma^2} = \sum_{k=0}^{\infty} \frac{(1/2\,\sigma^2 t^2)^k}{k!} = \sum_{k=0}^{\infty} \left(\frac{\sigma^2}{2}\right)^k \frac{t^{2k}}{k!}$$

$$= 1 + \frac{\sigma^2}{2} t^2 + \left(\frac{\sigma^2}{2}\right)^2 \frac{t^4}{2} + \left(\frac{\sigma^2}{2}\right)^3 \frac{t^6}{3!} + \cdots$$

Consider r odd. Since all the powers of t are even, r odd will result in an espression that has a t everywhere - i.e. there will be no constant. When we evaluate at 0, this will give $\left.\frac{\partial^r}{\partial t^r} e^{1/2 t^2 \sigma^2}\right|_{t=0} = 0$.

As an example,

$$r=1 \Rightarrow \left(\frac{\sigma^2}{2}\right) 2t + \left(\frac{\sigma^2}{2}\right)^2 \frac{4t^3}{2} + \left(\frac{\sigma^2}{2}\right)^3 \frac{6t^5}{3!} + \cdots \Big|_{t=0} = 0$$

$$r=2 \Rightarrow \left(\frac{\sigma^2}{2}\right) 2 + \left(\frac{\sigma^2}{2}\right)^2 \frac{4\cdot 3 t^2}{2} + \left(\frac{\sigma^2}{2}\right)^3 \frac{6\cdot 5 t^4}{3!} + \cdots \Big|_{t=0} = \sigma^2$$

$$r=3 \Rightarrow 0 + \left(\frac{\sigma^2}{2}\right)^2 \frac{4\cdot 3\cdot 2\, t}{2} + \left(\frac{\sigma^2}{2}\right)^3 \frac{6\cdot 5\cdot 4\, t^3}{3!} + \cdots \Big|_{t=0} = 0$$

$$r=4 \Rightarrow 0 + \left(\frac{\sigma^2}{2}\right)^2 \frac{4\cdot 3\cdot 2}{2} + \left(\frac{\sigma^2}{2}\right)^3 \frac{6\cdot 5\cdot 4\cdot 3\, t^2}{3!} + \cdots \Big|_{t=0} = 3\sigma^4$$

$$r=5 \Rightarrow 0 + 0 + \left(\frac{\sigma^2}{2}\right)^3 \frac{6!\, t}{3!} + \cdots \Big|_{t=0} = 0$$

$$r=6 \Rightarrow 0 + 0 + \left(\frac{\sigma^2}{2}\right)^3 \frac{6!}{3!} + \cdots \Big|_{t=0} = \left(\frac{\sigma^2}{2}\right)^3 \frac{6!}{3!}$$

4. (c) So, r odd $\Rightarrow E(X-\mu)^r = 0$

r even gives

$$E(X-\mu)^r = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!}$$

We can simplify this expression:

$$\left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!} = (\sigma^r)\left(\frac{1}{2}\right)^{r/2} \frac{r!}{(r/2)(r/2-1)(r/2-2)\cdots 1}$$

$$= (\sigma^r)\left(\frac{1}{2}\right)^{r/2} \frac{r!}{(r/2)\left(\frac{r-2}{2}\right)\left(\frac{r-4}{2}\right)\cdots\left(\frac{2}{2}\right)}$$

$$= (\sigma^r)(1/2)^{r/2} \frac{r!}{(1/2)^{r/2}\, r(r-2)(r-4)\cdots 2}$$

$$= (\sigma^r) \frac{r(r-1)(r-2)\cdots 1}{r(r-2)(r-4)\cdots 2} = \left[(r-1)(r-3)(r-5)\cdots 1\right]\sigma^r$$

$$\Rightarrow E(X-\mu)^r = \begin{cases} 0 & \text{if } r \text{ odd} \\ \left[(r-1)(r-3)\cdots 1\right]\sigma^r & \text{if } r \text{ even} \end{cases}$$

(d) $E(e^X) = M_X(1) = e^{\mu + \sigma^2/2}$

5. $f_{X,Y}(x,y) = \begin{cases} \lambda e^{-\lambda y}\sqrt{\frac{\lambda}{\pi}} & x^2 < y < \infty,\ -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases}$

(a) $f_X(x) = \int_{x^2}^{\infty} f_{X,Y}(x,y)\,dy = \int_{x^2}^{\infty} \lambda e^{-\lambda y}\sqrt{\frac{\lambda}{\pi}}\,dy$

$= \left[-e^{-\lambda y}\sqrt{\frac{\lambda}{\pi}}\right]_{x^2}^{\infty} = 0 + e^{-\lambda x^2}\sqrt{\frac{\lambda}{\pi}}$

$= \sqrt{\frac{\lambda}{\pi}}\, e^{-\lambda x^2}$ for $-\infty < x < \infty$

(b) $f_{Y|X}(y|x) = f_{X,Y}/f_X = \begin{cases} \lambda e^{-\lambda(y-x^2)} & x^2 < y < \infty,\ -\infty < x < \infty \\ 0 & \text{otherwise} \end{cases}$

6. $P(\text{head}) = u$ $\quad P(\text{head}) = w$

$p_i = P(i \text{ heads occur})$ $\quad i = 0, 1, 2$

Can we find u, w s.t. $p_0 = p_1 = p_2$?

$p_0 = p_2 \iff uw = (1-u)(1-w) \iff$

$uw = 1 - w - u + uw \iff w + u = 1 \iff \begin{matrix} u = 1-w \\ w = 1-u \end{matrix}$

$p_1 = p_2 \iff u(1-w) + (1-u)w = uw \iff$

$(1-w)^2 + w^2 = (1-w)w \iff$

$1 - 2w + w^2 + w^2 = w - w^2 \iff$

$3w^2 - 3w + 1 = 0 \iff$

$w = \frac{3 \pm \sqrt{9-12}}{6}$. There's no $w \in [0,1]$ that solves this equation

So $p_0 = p_1 = p_2$ is not possible $\boxtimes$.

7. Suppose $X > Y$.

$$(X \wedge Y) + (X \vee Y) = Y + X$$

Suppose $X < Y$

$$(X \wedge Y) + (X \vee Y) = X + Y$$

Suppose $X = Y$

$$(X \wedge Y) + (X \vee Y) = X + X = Y + Y = X + Y$$

So, $X + Y = (X \wedge Y) + (X \vee Y)$

Now, $E(X+Y) = E(X) + E(Y)$

$E(X+Y) = E\big((X \wedge Y) + (X \vee Y)\big) = E(X \wedge Y) + E(X \vee Y)$

Equate two equations $\Rightarrow$

$$E(X \vee Y) = E(X) + E(Y) - E(X \wedge Y) \quad \blacksquare.$$

8. $\operatorname{Cov}(X,Y) = E(\operatorname{Cov}(X,Y|Z)) + \operatorname{Cov}(E(X|Z), E(Y|Z))$

$$\begin{aligned}
\operatorname{Cov}(X,Y) &= E(XY) - EX\,EY \\
&= E(XY) - E(E(X|Z)E(Y|Z)) + E(E(X|Z)\,E(Y|Z)) \\
&\qquad - EX\,EY \\
&= E[E(XY|Z)] - E[E(X|Z)E(Y|Z)] \\
&\qquad + E[E(X|Z)E(Y|Z)] - E(E(X|Z))\,E(E(Y|Z)) \\
&= \underbrace{E\{E(XY|Z) - E(X|Z)E(Y|Z)\}} + \underbrace{E\{E(X|Z)E(Y|Z)\} - E(E(X|Z))E(E(Y|Z))} \\
&= E(\operatorname{Cov}(X,Y|Z)) + \operatorname{Cov}(E(X|Z), E(Y|Z))
\end{aligned}$$

14.381 Final Examination
Fall, 2000

Instructions: This is a closed book exam. You have 180 minutes to answer the questions. Good luck!

1. Let X be a $N(0, \sigma^2)$ random variable conditional on σ^2 with conditional density function
$$f_{X|\sigma^2}(x|\sigma^2) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}\exp(-\frac{1}{2}x^2/\sigma^2).$$
Let σ^2 have a marginal distribution of a χ_1^2 with density function
$$f_{\sigma^2}(z) = \frac{1}{\sqrt{2}\Gamma(1/2)}z^{-1/2}\exp(-\frac{z}{2}).$$
   a) Find $E(X|\sigma^2)$ and $E(X)$.
   b) Find $E(X^2|\sigma^2)$ and $E(X^2)$.

   (Hint: use known facts about the normal distribution).

2. Let X and Y be random variables with a joint density $f_{X,Y}(x, y)$.
   a) Show that $Var(X) \geq Var(E(X|Y))$.
   b) Show that $Var(X) \geq E(Var(X|Y))$.
   c) Show that $Var(X) = Var(E(X|Y)) + E(Var(X|Y))$.

3. Let $e = (1, 0, 1, 0, ...)'$ be an $n \times 1$ vector and $a_n = n/2$ if n is even and $a_n = (n+1)/2$ when n is odd. Define $M = I_n - ee'/a_n$.
   a) Show that M is symmetric and idempotent.
   b) Find the rank of M. State clearly how you obtain the result.
   c) Let X be an $n \times 1$ vector of *iid* standard normal random variables with density $(2\pi)^{-1/2}\exp(-\frac{1}{2}x_i^2)$ for the i-th element of X. Find the distribution of $X'MX$.

4. Let $X_t, t = 1, ..., 2n$ be a sample drawn from a distribution such that
$$\begin{pmatrix} X_{2i-1} \\ X_{2i} \end{pmatrix} \sim N(0, \Sigma)$$
where
$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$
for $|\rho| \leq 1$ and $i = 1, ..., n$. Assume that (X_{2i-1}, X_{2i}) is independent of (X_{2j-1}, X_{2j}) for all $i \neq j$.

1\. a) $E(X|\sigma^2) = \boxed{0}$ (Since $X|\sigma^2 \sim N(0, \sigma^2)$)

$E(X) = E(E(X|\sigma^2)) = E(0) = \boxed{0}$

b) $E(X^2|\sigma^2) = Var(X|\sigma^2) + E(X|\sigma^2)^2$

$= \sigma^2 + 0 = \boxed{\sigma^2}$

$E(X^2) = E(E(X^2|\sigma^2)) = E(\sigma^2) = \boxed{1}$

(since if $Y \sim \chi^2_n$, $E(Y) = n$)

2\. $Var(X) = E(X^2) - E(X)^2$

$= E(E(X^2|Y)) - E(E(X|Y))^2$ (By Iterated Expect.

$= E(E(X^2|Y)) - E(E(X|Y)^2)$

$+ E(E(X|Y)^2) - E(E(X|Y))^2$

$= E(Var(X|Y))$

$+ Var(E(X|Y))$

(a) Since $Var(E(X|Y)) \geq 0$, we are done

(b) Since $Var(X|Y) \geq 0 \Rightarrow E(Var(X|Y)) \geq 0$, we are done

(c) Already shown

3. (a) $M = I_n - ee'/a_n$

- symmetric:

$$M' = I_n' - 1/a_n (ee')' = I_n - 1/a_n ee' = M$$

- idempotent:

$$MM = (I_n - ee'/a_n)(I_n - ee'/a_n)$$

$$= I_n - ee'/a_n - ee'/a_n + \frac{ee'ee'}{a_n^2}$$

$$\left\{ e'e = \begin{cases} n/2 & n \text{ even} \\ (n+1)/2 & n \text{ odd} \end{cases} = a_n \right\}$$

$$= I_n - 2ee'/a_n + ee'/a_n = I_n - ee'/a_n = M$$

(b) The rank of a symmetric, idempotent matrix equals its trace.

$$\text{rank}(M) = tr(M) = tr(I_n - ee'/a_n)$$

$$= tr(I_n) - tr(ee'/a_n)$$

$$= n - tr(e'e/a_n)$$

$$= n - tr(1) = \boxed{n-1}$$

3. (c) General Result: If A is an $n \times n$ symmetric, idempotent matrix and $X \sim N(0, I_n)$, then

$$X'AX \sim \chi^2_k \quad \text{where } \operatorname{rank}(A) = k$$

(See Pg. 4 of Section 8 Notes).

$$\Rightarrow \quad \boxed{X'MX \sim \chi^2_{n-1}}$$

4. (a) $\frac{1}{n}\sum_{t=1}^{2n} X_t^2 = a + o_p(1)$

To apply the WLLN, we need independent data.

$$\frac{1}{n}\sum_{t=1}^{2n} X_t^2 = \underbrace{\frac{1}{n}\sum_{t=1}^{n} X_{2t-1}^2}_{\text{sum of indep data}} + \underbrace{\frac{1}{n}\sum_{t=1}^{n} X_{2t}^2}_{\text{sum of indep data.}}$$

Applying the WLLN to each of the two RHS terms, we have

$$\frac{1}{n}\sum_{t=1}^{n} X_{2t-1}^2 \xrightarrow{p} E(X_i^2) = 1$$

$$\frac{1}{n}\sum_{t=1}^{n} X_{2t}^2 \xrightarrow{p} E(X_i^2) = 1$$

$$\frac{1}{n}\sum_{t=1}^{2n} X_t^2 = (1 + o_p(1)) + (1 + o_p(1))$$

$$= 2 + o_p(1) \quad \Rightarrow \quad \boxed{a = 2}$$