14.381 STATISTICS
FALL 2004

PRACTICE PROBLEMS

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Midterm Examination

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Carefully derive or explain all your solutions. You can use known facts about distributions established in class without proof, but make sure to explain why such facts apply in your case. Good luck!

1. a) Let $A$ and $B$ be events with positive probability. Show that $P(A^c | B) = 1 - P(A | B)$.
   b) Let $A_1, ..., A_n$ be events. Show that
   \[ P(\cap_{i=1}^n A_i) \geq \sum_{i=1}^n P(A_i) - (n - 1). \]

2. Let $X$ and $Y$ be random variables such that $EX^2 < \infty$ and $EY^2 < \infty$. Assume that there exist functions $h(Y)$ and $g(Y)$ such that $E(h(Y)^2) < \infty$ and $E(g(Y)^2) < \infty$. Define $f(Y, a) = ag(Y) + h(Y)$ where $a$ is some constant. Assume that $E(g(Y)(X - h(Y))) \neq 0$.
   a) Find the constant $a$ such that
   \[ a = \arg \min_a E(X - f(Y, a))^2. \]
   b) Show that $h(Y)$ cannot be the conditional expectation of $X$, i.e. show that $h(Y) \neq E(X | Y)$ by showing that there exists some function $\tilde{h}(Y)$ such that $E(X - \tilde{h}(Y))^2 < E(X - h(Y))^2$.

3. Let $X = [X_1, X_2]'$ have a bivariate normal distribution with parameters $\mu = 0$ and
   \[ \Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}. \]
   The density of $X$ is $f_X(x) = (2\pi)^{-1} (\det \Sigma)^{-1/2} \exp(-1/2x'\Sigma^{-1}x)$. Find expressions for
   a) $Ee^{t'X}$ where $t' = [t_1, t_2]$ and $t_1$ and $t_2$ are constants.
   b) $EX_1 X_2$.
   c) $EX_1^2$.
   d) $EX_1 X_2^2$.

4. Let $X$ be a $n \times 1$ vector of random variables with $EX = \mu$ and $E(X - \mu)(X - \mu)' = \Sigma$. Note that $X$ is not assumed to be multivariate normal. Let $a$ and $b$ be $n \times 1$ vectors and $c$ a constant scalar. Assume that $a'b = 0$. Let $Y = a'X + c$ and $Z = b'X$
   a) Find the mean and variance of $Y = a'X + c$.
   b) Find $\text{cov}(Y, Z)$. Are $Y$ and $Z$ independent?
5. The joint density of \( X \) and \( Y \) is given by

\[
    f_{X,Y}(x,y) = \begin{cases} 
    \frac{x^{\frac{3}{2}}y^{-\frac{1}{2}}}{(2\pi)^\frac{3}{2}\Gamma(\frac{1}{2})} & \text{if } 0 < y < x < \infty \\
    0 & \text{elsewhere}
\end{cases}
\]

for constants \( r = 1, 2, \ldots \) and \( \Gamma(r) \) the Gamma function.

a) Find the marginal density of \( X \). Indicate the range on which the marginal density is defined.
b) Find the conditional density of \( Y \) given \( X = x \). Make sure to indicate the range on which the conditional density is defined.
c) Find \( EX \) and \( EY \).
Midterm Examination

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Good luck!

1. Let $A$ and $B$ be events with positive probability. Show the following:
   a) If $A$ and $B$ are independent then $A$ and $B^c$ are independent.
   b) If $P(A|B) < P(A)$ then $P(B|A) < P(B)$.

2. Prove or disprove the following: If $E(Y|X) = X$ and $E(X|Y) = Y$ and both $EX^2$ and $EY^2$ are finite then $P[X = Y] = 1$ (Hint: Show that $\text{Var}(X - Y) = 0$).

3. Let $X = [X_1, X_2]'$ have a bivariate normal distribution with parameters $\mu = 0$ and $\Sigma = \begin{bmatrix} \sigma_1^2 & \rho \sigma_1 \sigma_2 \\ \rho \sigma_1 \sigma_2 & \sigma_2^2 \end{bmatrix}$.

   a) Write down the marginal density of $X_1$ and the conditional density of $X_1$ given $X_2$. Find the first two (conditional) moments of the conditional distribution (Hint: you can use known properties of the normal distribution without proof).
   b) Let $\varepsilon(b) = X_1 - bX_2$. Find the value for the parameter $b$, expressed in terms of the parameters $\Sigma$, such that $E[\varepsilon(b)]^2$ is minimal.

4. Let $X$ be a $n \times 1$ vector of random variables with $EX = \mu$ and $E(X - \mu)(X - \mu)' = \Sigma$. Note that $X$ is not assumed to be multivariate normal.

   a) Let $A$ be a $n \times n$ matrix and $b$ an $n \times 1$ vector. Find the mean vector and covariance matrix of $Y = AX + b$.
   b) Show that $EX'AX = \text{tr}(A(\Sigma + \mu \mu'))$.

5. Let $X$ and $Y$ be random variables with joint density
   \[ f_{X,Y}(x,y) = \begin{cases} e^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \]
   Let $Z = X + Y$.
   
   a) Find the joint density of $Z$ and $X$ (Hint: be careful to state where the density is defined).
   b) Find the marginal density of $Z$ and the marginal density of $X$.
   c) Find the conditional density of $Z$ given $X = x$. 

1. (a) \( P(A \cap B^c) = P(A \cap \Omega \setminus B) = P(A \cap \Omega) - P(A \cap B) \)
\[ = P(A) - P(A)P(B) = P(A)[1 - P(B)] \]
\[ = P(A)P(B^c) \]
(b) \( P(A \cap B) = P(A | B)P(B) = P(B | A)P(A) \)
\[ \Rightarrow \frac{P(B)}{P(B | A)} = \frac{P(A)}{P(A | B)} > 1 \Rightarrow P(B) > P(B | A) \]

2. \( \text{Var}(X-Y) = E(\text{Var}(X-Y | X)) + \text{Var}(E(X-Y | X)) \)
\[ = E(\text{Var}(X - X)) + \text{Var}(E(X - X)) \]
\[ = E(\text{Var}(0)) + \text{Var}(E(0)) = 0 + 0 = 0. \]
\( E(X-Y) = E(E(X-Y | X)) = E(E(X-X)) = E(0) = 0 \)
\( \Rightarrow X, Y \text{ have same expectation, their difference has 0 variance.} \Rightarrow \)
\( P(X = Y) = 1. \)
3. (a) \[ X_1 \sim N(\mu_1, \text{Var}(X_1)) = N(0, \sigma_1^2) \]
\[
 f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi} \sigma_1} e^{-\frac{1}{2} \left( \frac{x_1 - \mu_1}{\sigma_1} \right)^2}.
\]

\[ X_1 | X_2 \sim N(\mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (X_2 - \mu_2), \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}) \]
\[
 = N\left( \frac{\rho \sigma_1}{\sigma_2} X_2, \sigma_1^2 - \rho^2 \sigma_1^2 \right)
\]
\[
 = N\left( \frac{\rho \sigma_1}{\sigma_2} X_2, (1 - \rho^2) \sigma_1^2 \right)
\]
\[
 f_{X_1|X_2}(x_1|x_2) = \frac{1}{\sqrt{2\pi} (1 - \rho^2) \sigma_1^2} e^{-\frac{1}{2} \left( \frac{x_1 - \frac{\rho \sigma_1}{\sigma_2} X_2}{\sqrt{1 - \rho^2}} \right)^2}.
\]

\[ E(X_1 | X_2) = \frac{\rho \sigma_1}{\sigma_2} X_2 \]

\[ E(X_1^2 | X_2) = \text{Var}(X_1 | X_2) + E(X_1 | X_2)^2 \]
\[
 = \sigma_1^2 - \rho^2 \sigma_1^2 + \rho^2 \sigma_1^2 X_2^2 \]
\[
 = \sigma_1^2 \left[ 1 + \rho^2 \left( \frac{X_2^2}{\sigma_2^2} - 1 \right) \right]
\]

(b) \[ \arg \min_b E\left( X_1 - b X_2 \right)^2 = E\left( X_1^2 - 2 b X_1 X_2 + b^2 X_2^2 \right) \]
\[ \Rightarrow \arg \min_b E(X_1^2) - 2 b E(X_1 X_2) + b^2 E(X_2^2) \]

\[ \text{FDC: } -2 E(X_1 X_2) + 2 b E(X_2^2) = 0 \]
\[ \Rightarrow b = \frac{E(X_1 X_2)}{E(X_2^2)} = \frac{\text{Cov}(X_1, X_2)}{\text{Var}(X_2)} = \frac{\rho \sigma_1 \sigma_2}{\sigma_2^2} = \left( \frac{\rho \sigma_1}{\sigma_2} \right)^2 \]
(a) \[ Y = AX + b \]

\[ E(Y) = A E(X) + b = A \mu + b \]

\[ \text{Var}(Y) = A \text{Var}(X) A' = A \Sigma A' \]

(b) \[ \chi' A \chi \text{ is } 1 \times 1 \text{ so } \chi' A \chi = tr(\chi' A \chi) \]

- By the property of the trace, \( tr(\chi' A \chi) = tr(A \chi \chi') \)
  (cyclic property of trace matrices)

\[ \Sigma = E((X-\mu)(X-\mu')) = E(XX'-\mu \chi'-\chi \mu' + \mu \mu') \]

\[ = E(XX') - \mu \mu' - \mu \mu' + \mu \mu' = E(XX') - \mu \mu' \]

\[ \Rightarrow E(XX') = \Sigma + \mu \mu' \]

- Finally,

\[ E(Y'AX) = E(tr(\chi' A \chi)) \]

\[ = E(tr(A \chi \chi')) \]

\[ = tr(E(\chi \chi')) \]

\[ = tr(A E(\chi \chi')) \]

\[ = tr(A (\Sigma + \mu \mu')) \]
5. \[ f_{x,y}(x,y) = \begin{cases} e^{-(x+y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases} \]

\[ z = x + y \]

(a) \[ z = x + y \quad v = x \]

\[ \Rightarrow x = v \quad y = z - v \quad \Rightarrow \begin{cases} x > 0 \Rightarrow v > 0 \\ y > 0 \Rightarrow z - v > 0 \Rightarrow z > v \end{cases} \]

\[ J = \left| \begin{array}{cc} \frac{dx}{dv} & \frac{dx}{dz} \\ \frac{dy}{dv} & \frac{dy}{dz} \end{array} \right| = \left| \begin{array}{cc} 1 & 0 \\ -1 & 1 \end{array} \right| = 1 \]

\[ f_{z,v} = f_{x,y}(v, z - v) = e^{-(v + z - v)} = e^{-z} \]

\[ \Rightarrow f_{z,x} = \begin{cases} e^{-z} & 0 < x < z \\ 0 & \text{otherwise} \end{cases} \]

(b) \[ f_z = \int_0^z e^{-z} \, dz = -e^{-z} \bigg|_0^z = be^{-z} \quad \forall \quad z > 0 \]

\[ f_x = \int_0^\infty e^{-z} \, dz = -e^{-z} \bigg|_0^\infty = 0 + e^{-x} = e^{-x} \quad \forall \quad x > 0 \]

(c) \[ f_{z|x} = f_{x,z} / f_x = e^{-z} / e^{-x} = e^{x - z} \]

\[ = \begin{cases} e^{-(z-x)} & 0 < x < z \\ 0 & \text{otherwise} \end{cases} \]
14.381 Midterm Examination
Fall, 1999

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Make sure to show all derivations needed for your results. Good luck!

1. Show that it is not possible to find events $A$, $B$, $C$ such that

$$P(A) = P(B) = P(C) = \frac{3}{8}$$

and

$$P(A \cup B) = P(A \cup C) = P(B \cup C) = \frac{3}{4}. $$

2. Show that for two random variables $X$ and $Y$ such that $E |X|^2 < \infty$, $E |Y|^2 < \infty$ it follows that

$$\text{var}(E(X \mid Y)) \leq \text{var}(X).$$

3. Let $X$ be a normal random variable with density $f_X(x) = (2\pi)^{-1/2} \sigma^{-1} e^{-1/2(x-\mu)/\sigma^2}$ where $\mu = EX$ and $\sigma^2 = \text{var}(X)$. Let $K_i$ be the $i$-th cumulant of $X$. Let $Y = \exp(X)$.

Find

a) $EY$,

b) $EY^2$,

c) $EY^r$, $r > 0$,

d) $K_1$,

e) $K_2$,

f) $K_i$ $i \geq 3$, $i$ integer

4. Let $X_1, \ldots, X_N$ be a collection of mutually independent random variables with $X_i \sim N(0, \sigma_i)$ where $\sigma_i$ is a $\chi^2_{(i)}$ random variable. Define $S_N = \sum_{i=1}^{N} X_i$ with $N$ a constant integer $N > 0$.

Find $E(S_N)$ and $\text{var}(S_N)$ (Hint: you can use the fact that the moment generating function for the $\chi^2_{(i)}$ distribution is $1/(1-2t)$ to obtain the moments for $\sigma_i$).

5. Let $X_1$ and $X_2$ be independent random variables with densities

$$f_{X_i}(x) = \begin{cases} \frac{\lambda^n_i x^{n_i-1}}{\Gamma(n_i)} e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } n_i > 0, \lambda > 0, i = 1, 2$$

Define $Y_1 = X_1 + X_2$ and $Y_2 = X_1/X_2$.

a) Find the joint density of $Y_1$ and $Y_2$.

b) Show that $Y_1$ and $Y_2$ are independent.
Practice Problems - No Due Date

1. (Midterm 1999) Let $X$ be a normal random variable with density

$$f_X(x) = (2\pi)^{-1/2} \sigma^{-1} e^{-1/2(x-\mu)/\sigma^2}$$

Define $\mu = E(X)$, $\sigma^2 = Var(X)$, $Y = \exp(X)$. Let $K_i$ be the $i$-th cumulant of $X$. Evaluate:

(a) $E(Y)$
(b) $E(Y^2)$
(c) $E(Y^r)$, $r > 0$
(d) $K_1$
(e) $K_2$
(f) $K_i$, $i \geq 3$, $i$ integer

2. (Waiver 1999) Given $N$, let $X_1, \ldots, X_N$ be an iid sample of normal random variables with mean $\mu$ and variance $\sigma^2$. Let $N$, the sample size, be a Poisson random variable

$$P(N = n) = \frac{e^{-\lambda} \lambda^n}{n!} \text{ for } n = 0, 1, 2, \ldots, \lambda > 0$$

Consider the sum of the observations

$$S = \left\{ \begin{array}{ll}
\sum_{i=1}^{N} X_i & N > 0 \\
0 & N = 0
\end{array} \right\}$$

Find $ES$ and $ES^2$

(Hint: you can use the fact that $EN = \lambda$ and $Var(N) = \lambda$ without proof).

3. (Midterm 1998) We observe a process of events occurring randomly in time. Let $N_t$ be the total number of events between 0 and $t$. $N_t$ is distributed as a Poisson random variable with parameter $\lambda$, i.e.

$$P(N_t = k) = \frac{e^{-\lambda t} (\lambda t)^k}{k!}$$

Also assume that $N_t$ and $N_{t+s} - N_t$ are independent for all $t, s > 0$ and that $N_{t+s} - N_t$ is Poisson with $P(N_{t+s} - N_t = k) = \frac{e^{-\lambda s} (\lambda s)^k}{k!}$
(a) Find $E N_t$ for some $t$ fixed.
(b) Find $E(N_{t+s} | N_t)$ for $t, s > 0$ fixed.

4. (Midterm 1998) Assume that $X$ is a normal r. v. with moment generating function $e^{t \mu + \frac{1}{2} t^2 \sigma^2}$. Find the following moments:

(a) $E(X^2)$
(b) $E(X^3)$
(c) $E((X - \mu)^r)$ for $r > 0$; $r$ integer
(d) $E(e^X)$

6. (Midterm 1998) If the joint density of $(X, Y)$ is

$$f_{X,Y}(x, y) = \left\{ \begin{array}{ll}
\lambda e^{-\lambda y} \sqrt{\frac{1}{\pi}} & x^2 < y < \infty, \ -\infty < x < \infty \\
0 & \text{elsewhere}
\end{array} \right.$$

Find:

(a) The marginal density of $X$
(b) The conditional density of $Y$ given $X = x$

6. Two pennies, one with $P(\text{head}) = u$ and one with $P(\text{head}) = w$, are to be tossed together independently. Define $p_0 = P(0 \text{ heads occur})$, $p_1 = P(1 \text{ head occurs})$, $p_2 = P(2 \text{ heads occur})$. Can $u$ and $w$ be chosen such that $p_0 = p_1 = p_2$? Prove your answer.

6. Betteley (1977) provides an interesting addition law of expectations. Let $X$ and $Y$ be any two random variables and define $X \wedge Y = \min(X, Y)$ and $X \vee Y = \max(X, Y)$. Analogous to the probability law $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, show that $E(X \vee Y) = EX + EY - E(X \wedge Y)$.

(Hint: Establish that $X + Y = (X \wedge Y) + (X \vee Y)$)

8. For any three random variables $X, Y,$ and $Z$ with finite variances, prove the covariance identity $\text{Cov}(X, Y) = E(\text{Cov}(X, Y | Z)) + \text{Cov}(E(X | Z), E(Y | Z))$. 

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Problem Set 6

(Practice Problems)

Solutions
\[ f_x(x) = \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{1}{2} \left( \frac{x - \mu}{\sigma} \right)^2}, \quad y = e^x \]

(a) \[ E(y) = E(e^x) = M_x(1) = e^{\mu t + \frac{1}{2} t^2 \sigma^2} \bigg|_{t=1} = e^{\mu + \frac{1}{2} \sigma^2} \]

(b) \[ E(y^2) = E(e^{2x}) = M_x(2) = e^{2\mu + 2\sigma^2} \]

(c) \[ E(y^r) = E(e^{rx}) = M_x(r) = e^{r\mu + \frac{1}{2} r^2 \sigma^2} \]

(d) \[ K_1 = \frac{\partial}{\partial t} K_x(t) \bigg|_{t=0} \]

\[ K_x(t) = \log (M_x(t)) = \log \left( e^{\mu t + \frac{1}{2} t^2 \sigma^2} \right) = \mu t + \frac{1}{2} t^2 \sigma^2 \]

\[ K_1 = (\mu + t \sigma^2) \bigg|_{t=0} = \mu \]

(e) \[ K_2 = \frac{\partial^2}{\partial t^2} K_x(t) \bigg|_{t=0} = \sigma^2 \bigg|_{t=0} = \sigma^2 \]

(f) \[ K_i, \quad i \geq 3, \quad i \text{ integer} \]

\[ K_i = \frac{\partial^i}{\partial t^i} K_x(t) \bigg|_{t=0} = 0 \quad \forall \ i \geq 3, \ i \text{ integer.} \]
1. \( X_i \mid N \sim N(\mu, \sigma^2) \) \( \forall i \)

\( X_i \mid N \) independent across observations

\( P(N=n) = \left( e^{-\lambda} \frac{\lambda^n}{n!} \right) / n! \)

\( S = \begin{cases} \sum_{i=1}^{\infty} X_i & \text{if } N > 0 \\ 0 & \text{if } N = 0 \end{cases} \)

\[
E(S) = E\left( \sum_{i=1}^{\infty} X_i \right) = E\left( E\left( \sum_{i=1}^{\infty} X_i \mid N \right) \right) \\
= E\left( \sum_{i=1}^{\infty} E(X_i \mid N) \right) = E(N\mu) = \mu E(N) = \mu \lambda
\]

\[
E(S^2) = Var(S) + (E(S))^2
\]

\[
Var(S) = Var \left( \sum_{i=1}^{\infty} X_i \right) = Var \left( E \left( \sum_{i=1}^{\infty} X_i \mid N \right) \right) + E \left( Var \left( \sum_{i=1}^{\infty} X_i \mid N \right) \right) \\
= Var(N\mu) + E \left( \sum_{i=1}^{\infty} Var(X_i \mid N) \right) \\
= \mu^2 Var(N) + E \left( N \sigma^2 \right) \quad \text{because } X_i \text{ are iid} \\
= \mu^2 \lambda + \sigma^2 \lambda
\]

\[
E(S^2) = \mu^2 \lambda + \sigma^2 \lambda + \mu^2 \lambda^2 = \lambda \left( \mu^2 (1 + \lambda) + \sigma^2 \right)
\]
3. \( P(N_t = k) = \left( e^{-\lambda t} (\lambda t)^k \right) / k! \)

\( N_t, N_{t+s} - N_t \) independent \( \forall t, s > 0 \),

\( N_{t+s} - N_t \) is Poisson: \( P(N_{t+s} - N_t = k) = e^{-\lambda s} (\lambda s)^k / k! \)

(a) \( E(N_t) = \sum_{k=0}^{\infty} k e^{-\lambda t} (\lambda t)^k / k! \)

\[= \sum_{k=1}^{\infty} e^{-\lambda t} (\lambda t)^k / (k-1)! = \lambda t \sum_{k=1}^{\infty} e^{-\lambda t} (\lambda t)^{k-1} = \lambda t \]

(b) \( E(N_{t+s} | N_t) = E(N_{t+s} - N_t | N_t) + E(N_t | N_t) \)

\[= E(N_{t+s} - N_t) + E(N_t | N_t) \]

by independence

\[= \lambda s + N_t \]

using result from (a).

\( E(N_{t+s} | N_t) \) is a random variable - it is a function of \( N_t \).
(a) \( E(X^2) = M_X^{(1)}(0) = \frac{\partial^2 M_X(t)}{\partial t^2} \bigg|_{t=0} \)

\[
= \left. \frac{\partial}{\partial t} \left( e^{tm + \frac{1}{2} \sigma^2 t^2} (m + t \sigma^2) \right) \right|_{t=0} \\
= e^{tm + \frac{1}{2} \sigma^2 t^2} (m + t \sigma^2)^2 \bigg|_{t=0} \\
= m^2 + \sigma^2
\]

(b) \( E(X^3) = 3 e^{tm + \frac{1}{2} \sigma^2 t^2} ((m + t \sigma^2)^2 + \sigma^2) \bigg|_{t=0} \)

\[
= e^{tm + \frac{1}{2} \sigma^2 t^2} ((m + t \sigma^2)^2 + (m + t \sigma^2) \sigma^2 + 2(m + t \sigma^2) \sigma^2) \bigg|_{t=0} \\
= e^{tm + \frac{1}{2} \sigma^2 t^2} \left[ (m + t \sigma^2)^3 + 3(m + t \sigma^2) \sigma^2 \right] \bigg|_{t=0} \\
= m^3 + 3m \sigma^2
\]

(c) \( E(X - \mu)^r \) for \( r > 0 \), \( r \) integer

Define \( Y = X - \mu \)

\[
M_Y(t) = E(e^{yt}) = E(e^{(X-\mu)t}) = e^{-\mu t} E(e^{xt}) \\
= e^{-\mu t} e^{tu + \frac{1}{2} t^2 \sigma^2} = e^{\frac{1}{2} t^2 \sigma^2}
\]

\[
E(Y^r) = E(Y^r) = \frac{\partial^r}{\partial t^r} \bigg|_{t=0} e^{\frac{1}{2} t^2 \sigma^2}
\]
1. (c) We want to evaluate the \( r \)th derivative of \( e^{\frac{1}{2} t^2 \sigma^2} \) at \( t = 0 \).

\[
e^{\frac{1}{2} t^2 \sigma^2} = \sum_{k=0}^{\infty} \frac{(\frac{1}{2} \sigma^2 t^2)^k}{k!} = \sum_{k=0}^{\infty} \left( \frac{\sigma^2}{2} \right)^k \frac{t^{2k}}{k!}
\]

\[
= 1 + \frac{\sigma^2}{2} t^2 + \left( \frac{\sigma^2}{2} \right)^2 \frac{t^4}{2} + \left( \frac{\sigma^2}{2} \right)^3 \frac{t^6}{3!} + \ldots
\]

Consider \( r \) odd. Since all the powers of \( t \) are even, \( r \) odd will result in an expression that has a \( t \) everywhere — i.e., there will be no constant. When we evaluate at \( 0 \), this will give \( \frac{d^r}{dt^r} e^{\frac{1}{2} t^2 \sigma^2} \bigg|_{t=0} = 0 \).

As an example,

\( r = 1 \) \( \Rightarrow \) \( \left( \frac{\sigma^2}{2} \right) 2t + \left( \frac{\sigma^2}{2} \right)^2 \frac{4t^3}{2} + \left( \frac{\sigma^2}{2} \right)^3 \frac{6t^5}{3!} + \ldots \bigg|_{t=0} = 0 \)

\( r = 2 \) \( \Rightarrow \) \( \left( \frac{\sigma^2}{2} \right)^2 2 + \left( \frac{\sigma^2}{2} \right)^2 \frac{4t^4}{2} + \left( \frac{\sigma^2}{2} \right)^3 \frac{6t^6}{3!} + \ldots \bigg|_{t=0} = \sigma^2 \)

\( r = 3 \) \( \Rightarrow \) \( 0 + \left( \frac{\sigma^2}{2} \right)^2 \frac{4t^3}{2} + \left( \frac{\sigma^2}{2} \right)^3 \frac{6t^5}{3!} + \ldots \bigg|_{t=0} = 0 \)

\( r = 4 \) \( \Rightarrow \) \( 0 + \left( \frac{\sigma^2}{2} \right)^2 \frac{4t^4}{2} + \left( \frac{\sigma^2}{2} \right)^3 \frac{6t^6}{3!} + \ldots \bigg|_{t=0} = 3 \sigma^4 \)

\( r = 5 \) \( \Rightarrow \) \( 0 + 0 + \left( \frac{\sigma^2}{2} \right)^3 \frac{6t^5}{3!} + \ldots \bigg|_{t=0} = 0 \)

\( r = 6 \) \( \Rightarrow \) \( 0 + 0 + \left( \frac{\sigma^2}{2} \right)^3 \frac{6^1 t^6}{3!} + \ldots \bigg|_{t=0} = \left( \frac{\sigma^2}{2} \right)^3 \frac{6^1}{3!} \)
Y. (c) So, \( r \) odd \( \Rightarrow E(x-\mu)^r = 0 \)

\[ \text{r even gives} \]

\[ E(x-\mu)^r = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!} \]

We can simplify this expression,

\[ \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!} = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!} \frac{r!}{(r/2)(r/2-1)(r/2-2) \cdots 1} \]

\[ = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)!} \frac{r!}{(r/2)(r/2-1)(r/2-2) \cdots \left(\frac{r}{2}\right)} \]

\[ = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r!}{(r/2)^{r/2} \cdot r \cdot (r-2) \cdot (r-4) \cdots 2} \]

\[ = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{r \cdot (r-1) \cdot (r-2) \cdots 1}{r \cdot (r-2) \cdot (r-4) \cdots 2} = \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{(r-1)(r-3)(r-5) \cdots 1}{2} \]

\[ \Rightarrow E(x-\mu)^r = \begin{cases} 0 & \text{if } r \text{ odd} \\ \left(\frac{\sigma^2}{2}\right)^{r/2} \frac{(r-1)(r-3)(r-5) \cdots 1}{2} \sigma^r & \text{if } r \text{ even} \end{cases} \]

(d) \( E(e^x) = M_x(1) = e^{\mu + \sigma^2/2} \)
5. \( f_{X,Y}(x,y) = \begin{cases} \lambda e^{-\lambda y} \sqrt{\frac{2}{\pi}} & x^2 < y < \infty, -\infty < x < \infty \\ 0 & \text{elsewhere} \end{cases} \)

(a) \( f_x(x) = \int_{x^2}^{\infty} f_{X,Y}(x,y) \, dy = \int_{x^2}^{\infty} \lambda e^{-\lambda y} \sqrt{\frac{2}{\pi}} \, dy \)

\[
= \left[ -e^{-\lambda y} \sqrt{\frac{2}{\pi}} \right]_{x^2}^{\infty} = 0 + e^{-\lambda x^2} \sqrt{\frac{2}{\pi}} \]

\[= \sqrt{\frac{2}{\pi}} e^{-\lambda x^2} \quad \text{for} \quad -\infty < x < \infty \]

(b) \( f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_x(x)} = \begin{cases} \lambda e^{-\lambda(y-x^2)} & x^2 < y < \infty \\ 0 & \text{otherwise} \end{cases} \)

6. \( P(\text{head}) = u \quad P(\text{head}) = w \)

\( p_i = P(\text{i heads occur}) \quad i = 0, 1, 2 \)

Can we find \( u, w \) s.t. \( p_0 = p_1 = p_2 \)?

\( p_0 = p_2 \iff u w = (1-u)(1-w) \iff u w = 1-w-u+u w \iff w+u = 1 \iff u = 1-w \quad w = 1-u \)

\( p_1 = p_2 \iff u (1-w) + (1-u) w = u w \iff (1-w)^2 + w^2 = (1-w)w \iff 1-2w+w^2+w^2 = w-w^2 \iff 3w^2-3w+1 = 0 \iff w = \frac{3 \pm \sqrt{9-12}}{6} \in [0,1] \)

That solves this equation.

So \( p_0 = p_1 = p_2 \) is not possible \( \Box \).
Suppose \( X > Y \).
\[
(X \land Y) + (X \lor Y) = Y + X
\]
Suppose \( X < Y \)
\[
(X \land Y) + (X \lor Y) = X + Y
\]
Suppose \( X = Y \)
\[
(X \land Y) + (X \lor Y) = X + X = Y + Y = X + Y
\]
So, \( X + Y = (X \land Y) + (X \lor Y) \)

Now, \( E(X+Y) = E(X) + E(Y) \)
\[
E(X+Y) = E((X \land Y) + (X \lor Y)) = E(X \land Y) + E(X \lor Y)
\]

Equate the equations \( \Rightarrow \)
\[
E(X \lor Y) = E(X) + E(Y) - E(X \land Y) \quad \square
\]

8. \( \text{Cov}(X,Y) = E(\text{Cov}(X,Y_{12})) + \text{Cov}(E(X_{12}), E(Y_{12})) \)

\[
\text{Cov}(X,Y) = E(XY) - EXEY
\]
\[
= E(XY) - E(E(X_{12})E(Y_{12})) + E(E(X_{12})E(Y_{12}))
\]
\[
- EXEY
\]
\[
= E[EX_{12}] - E[E(X_{12})E(Y_{12})]
\]
\[
+ E[E(X_{12})E(Y_{12})] = E(EX_{12})E(Y_{12})
\]
\[
= E \left( E(X_{12}) - E(X_{12})E(Y_{12}) \right)^2 + E \left( E(X_{12})E(Y_{12}) - E(E(X_{12})E(Y_{12})) \right)
\]
\[
= E(\text{Cov}(X,Y_{12})) + \text{Cov}(E(X_{12}), E(Y_{12}))
\]
14.381 Final Examination
Fall, 2000

Instructions: This is a closed book exam. You have 180 minutes to answer the questions. Good luck!

1. Let $X$ be a $N(0, \sigma^2)$ random variable conditional on $\sigma^2$ with conditional density function

$$f_{X|\sigma^2}(x|\sigma^2) = \frac{1}{\sqrt{2\pi} \sigma} \exp\left(-\frac{1}{2} \frac{x^2}{\sigma^2}\right).$$

Let $\sigma^2$ have a marginal distribution of a $\chi^2_1$ with density function

$$f_{\sigma^2}(\sigma) = \frac{1}{\sqrt{2\Gamma(1/2)}} \sigma^{-1/2} \exp\left(-\frac{\sigma}{2}\right).$$

(a) Find $E(X|\sigma^2)$ and $E(X)$. 
(b) Find $E(X^2|\sigma^2)$ and $E(X^2)$.

(Hint: use known facts about the normal distribution).

2. Let $X$ and $Y$ be random variables with a joint density $f_{X,Y}(x,y)$.

(a) Show that $\text{Var}(X) \geq \text{Var}(E(X|Y))$. 
(b) Show that $\text{Var}(X) \geq E(\text{Var}(X|Y))$. 
(c) Show that $\text{Var}(X) = E(\text{Var}(X|Y)) + E(\text{Var}(X|Y))$.

3. Let $\epsilon = (1,0,1,0,...)'$ be an $n \times 1$ vector and $a_n = n/2$ if $n$ is even and $a_n = (n+1)/2$ when $n$ is odd. Define $M = I_n - ee'/a_n$.

(a) Show that $M$ is symmetric and idempotent. 
(b) Find the rank of $M$. State clearly how you obtain the result. 

(c) Let $X$ be an $n \times 1$ vector of iid standard normal random variables with density $(2\pi)^{-1/2} \exp(-\frac{1}{2} x^2)$ for the $i$-th element of $X$. Find the distribution of $X'MX$.

4. Let $X_t, t = 1, ..., 2n$ be a sample drawn from a distribution such that

$$\begin{pmatrix} X_{2i-1} \\ X_{2i} \end{pmatrix} \sim N(0, \Sigma)$$

where

$$\Sigma = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

for $|\rho| \leq 1$ and $i = 1, ..., n$. Assume that $(X_{2i-1}, X_{2i})$ is independent of $(X_{2j-1}, X_{2j})$ for all $i \neq j$. 
1. a) \( E(X|\sigma^2) = \square \) (since \( X|\sigma^2 \sim N(0, \sigma^2) \))
   \[
   E(X) = E(E(X|\sigma^2)) = E(0) = \square
   \]
b) \( E(X^2|\sigma^2) = \text{Var}(X|\sigma^2) + E(X|\sigma^2)^2 \)
   \[
   = \sigma^2 + 0 = \sqrt{\sigma^2}
   \]
   \( E(X^2) = E(E(X^2|\sigma^2)) = E(\sigma^2) = \square \)
   (since if \( Y \sim X_n^2 \), \( E(Y) = n \))

2. \( \text{Var}(X) = E(X^2) - E(X)^2 \)
   \[
   = E(E(X^2|\sigma^2)) - E(E(X|\sigma^2))^2 \quad (b) \text{Ind. Exp.}
   \]
   \[
   = E(E(X^2|\sigma^2)) - E(E(X^2))^2
   \]
   \[
   + E(E(X)^2) - E(E(X))^2
   \]
   \[
   = E(\text{Var}(X|\sigma^2)) + \text{Var}(E(X|\sigma^2))
   \]
   (a) Since \( \text{Var}(E(X|\sigma^2)) \geq 0 \), we are done
   (b) Since \( \text{Var}(X|\sigma^2) \geq 0 \implies E(\text{Var}(X|\sigma^2)) \geq 0 \), we are done
   (c) Already shown
3. (a) \[ M = I_n - \frac{ee'}{\alpha_n} \]

- Symmetric:
  \[ M' = (I_n - \frac{ee'}{\alpha_n})(I_n - \frac{ee'}{\alpha_n})' = I_n - \frac{ee'}{\alpha_n} \quad ee' = M \]

- Idempotent:
  \[ MM = (I_n - \frac{ee'}{\alpha_n})(I_n - \frac{ee'}{\alpha_n}) \]
  \[ = I_n - \frac{ee'}{\alpha_n} - \frac{ee'}{\alpha_n} + \frac{ee'ee'}{\alpha_n^2} \]
  \[ \{ ee' = \begin{cases} \frac{n}{2} & n \text{ even} \\ \frac{(n+1)}{2} & n \text{ odd} \end{cases} = \frac{\alpha_n}{2} \} \]
  \[ = \frac{I_n}{2} - 2\frac{ee'}{\alpha_n} + \frac{ee'}{\alpha_n} - \frac{ee'ee'}{\alpha_n^2} = I_n - \frac{ee'}{\alpha_n} - M \]

(b) The rank of a symmetric, idempotent matrix equals its trace.

\[ \text{rank}(M) = \text{tr}(M) = \text{tr}(I_n - \frac{ee'}{\alpha_n}) \]
\[ = \text{tr}(I_n) - \text{tr}(ee'/\alpha_n) \]
\[ = n - \text{tr}(ee'/\alpha_n) \]
\[ = n - \text{tr}(1) = \lfloor n - 1 \rfloor \]
3. (c) General result: If \( A \) is an \( nxn \) symmetric, idempotent matrix and \( X \sim N(0, I_n) \), then

\[
X'AX \sim X^2 \quad \text{while rank} \ (A) = k
\]

(See Pg. 71 of Sect. 3 Notes).

\[\Rightarrow \quad X'MX \sim X^2_{n-1}\]

4. (a) \( \frac{1}{n} \sum_{i=1}^{2n} X_i^2 = \alpha + o_P(1) \)

To apply the WLLN, we need independent data.

\[\frac{1}{n} \sum_{i=1}^{2n} X_i^2 = \frac{1}{n} \sum_{i=1}^{m} X_{2i}^2 + \frac{1}{n} \sum_{i=1}^{m} X_{2i+1}^2 = \frac{m}{n} \sum_{i=1}^{m} X_{2i}^2 + \frac{m}{n} \sum_{i=1}^{m} X_{2i+1}^2
\]

sum of indep. data

sum of indep. data.

Applying the WLLN to each of the two RHS terms, we have

\[\frac{1}{n} \sum_{i=1}^{m} X_{2i}^2 \xrightarrow{p} E(X_{2i}^2) = 1\]

\[\frac{1}{n} \sum_{i=1}^{m} X_{2i+1}^2 \xrightarrow{p} E(X_{2i+1}^2) = 1\]

\[\frac{1}{n} \sum_{i=1}^{2n} X_i^2 = (1 + o_P(1)) + (1 + o_P(1))\]

\[= 2 + o_P(1) \quad \Rightarrow \quad \alpha = 2\]