14.381 Solutions Problem Set 0 (Set Theory)
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TA: José Tessada
(tessada@mit.edu)

A longer explanation is provided for the first exercise, showing more about the structure of these proofs; but, very detailed explanations are not necessary in general.

1. The proof provided here deals with each implication statement separately. If both are correct, then the double implication is done.

$$A \subseteq B \quad \Rightarrow \quad (A^c \cup B) = X$$

$$\begin{aligned} A \subseteq B \quad &\Rightarrow \quad x \in A \Rightarrow x \in B \\ \text{by definition} \quad & \quad\quad x \notin A \Rightarrow x \in A^c \\ &\Rightarrow \quad (A^c \cup B) = X \end{aligned}$$

and we know that $\forall x \in X$, either $x \in A$ or $x \notin A$. Given that, we have proved that any x belongs to at least one of the sets (B or A^c), then their union is equal to the universal set.

$$\begin{aligned} (A^c \cup B) \quad &= \quad X \Rightarrow A \subseteq B \\ &\Rightarrow \quad \forall x \in X, \text{ either } x \in A^c \text{ or } x \in B \text{ or both} \\ \text{if } x \in A \quad &\Rightarrow \quad x \notin A^c \Rightarrow x \in B \\ &\Rightarrow \quad A \subseteq B. \end{aligned}$$

2. For each of these equalities, you must show containment in both directions.

   (a) $x \in A \backslash B \Leftrightarrow x \in A$ and $x \notin B \Leftrightarrow x \in A$ and $x \notin A \cap B \Leftrightarrow x \in A \backslash (A \cap B)$. Also, $x \in A$ and $x \notin B \Leftrightarrow x \in A$ and x$\in B^c \Leftrightarrow x \in A \cap B^c$.

   (b) Suppose $x \in B$. Then either $x \in A$ or $x \in A^c$. If $x \in A$, then $x \in B \cap A$, and, hence $x \in (B \cap A) \cup (B \cap A^c)$. Thus $B \subset (B \cap A) \cup (B \cap A^c)$. Now suppose $x \in (B \cap A) \cup (B \cap A^c)$. then either $x \in (B \cap A)$ or $x \in (B \cap A^c)$. If $x \in (B \cap A)$, then $x \in B$. If $x \in (B \cap A^c)$, then $x \in B$. Thus $(B \cap A) \cup (B \cap A^c) \subset B$. Notice that if $M \subset N$ and $N \subset M$, then $M = N$, so we proved that $B = (B \cap A) \cup (B \cap A^c)$. Another way to proceed is to use the *Distributive Law* to solve the parenthesis in the statement.

   (c) Very similar to part a)
   $x \in B \backslash A \Leftrightarrow x \in B$ and $x \notin A \Leftrightarrow x \in B$ and $x \in A^c \Leftrightarrow x \in A \cap B^c$.

   (d) We can use the result from part b) to write:
   $A \cup B = A \cup [(B \cap A) \cup (B \cap A^c)] = A \cup (B \cap A) \cup A \cup (B \cap A^c) = A \cup [A \cup (B \cap A^c)] = A \cup (B \cap A^c)$.

3. (a) $x \in A \cup B \Leftrightarrow x \in A$ or $x \in B \Leftrightarrow x \in B \cup A$
   $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B \Leftrightarrow x \in B \cap A$.

   (b) $x \in A \cup (B \cup C) \Leftrightarrow x \in A$ or $x \in B \cup C \Leftrightarrow x \in A \cup B$ or $x \in C \Leftrightarrow x \in (A \cup B) \cup C$.
   (The proof for $A \cup (B \cup C) = (A \cup C) \cup B$ can be done using the same structure.)
   $x \in A \cap (B \cap C) \Leftrightarrow x \in A$ and $x \in B$ and $x \in C \Leftrightarrow x \in (A \cap B) \cap C$.

   (c) $x \in (A \cup B)^c \Leftrightarrow x \notin A$ or $x \notin B \Leftrightarrow x \in A^c$ and $x \in B^c \Leftrightarrow x \in A^c \cap B^c$
   $x \in (A \cap B)^c \Leftrightarrow x \notin A \cap B \Leftrightarrow x \notin A$ and $x \notin B$ (if it belongs to one but not to the other we are still ok!) $\Leftrightarrow x \in A^c$ or $x \in B^c \Leftrightarrow x \in A^c \cup B^c$.