14.381 Problem Set 1
Statistics Fall, 2004

TA: José Tessada (tessada@mit.edu)

Due September 18 in Recitation

1. Describe the sample space, event space, and probability measure for each of the following:
   (a) Measure the lifetime (in hours) of a particular brand of light bulb.
   (b) Two fair coins are tossed. The outcome is the total number of heads.
   (c) A student tosses a coin. If the coin shows heads, he does a micro problem set; if tails, he tosses a second coin. If the second coin shows heads, he does the micro problem set; if tails, he does a statistics problem set.

2. A committee is to consist of four academics and two industrialists, to be chosen from a larger group of seven academics (three of whom are women) and five industrialists (two of whom are women). For (c) and (d), assume that the four academics and two industrialists are chosen at random, with no restrictions as to the number of women chosen.
   (a) How many ways can the committee be formed?
   (b) How many ways can the committee be formed so that at least four members of the committee are women? (Hint: Solve for exactly 4 women, solve for exactly 5 women, then add.)
   (c) Susan is chosen to be a member of the committee. What is the probability that she is an academic?
   (d) Susan and Peter are both chosen to be members of the committee. What is the probability that they both are industrialists?

3. Let A, B and C be events such that $B \subset A$ and $P(C|A) = P(C|B)$. Show that
$$P(B|A \cap C) = P(B|A).$$

4. Show that it is not possible to find events A, B, C such that $P(A) = P(B) = P(C) = \frac{3}{8}$ and $P(A \cup B) = P(A \cup C) = P(B \cup C) = \frac{3}{4}$.

5. If $P(A) = \frac{1}{3}$ and $P(B^c) = \frac{1}{4}$, can A and B be disjoint? Can A and B be independent? If no, prove it; if yes, give an example.

6. John and Mary use a random device to decide who should wash dishes. They toss a coin five times. If at least four heads show up, John will do the dishes. If at least three tails show up, Mary will do the dishes. Otherwise, they will leave the dishes until tomorrow. The next day they repeat the procedure. However, if no one is assigned to wash dishes on the second day, John will give up and wash the dishes. Find the probability that
   (a) Mary has to do dishes two days in a row.
   (b) John will wash dishes tomorrow given that no one did it today.
   (c) Mary will wash dishes tomorrow given that no one did it today.
   (d) No one washed dishes on day 1 given that John washed them on day 2.

7. Three prisoners A, B, and C know that exactly two of them are going to be executed, but they do not know which two. Prisoner A knows that the jailer will not tell him whether or not he is going to be executed. He therefore asks the jailer to tell him the name of one prisoner other than A himself who will be executed. The jailer responds that B will be executed. Upon receiving this response, prisoner A reasons as follows: before he spoke to the jailer, the probability was $\frac{2}{3}$ that he will be executed. After speaking to the jailer, he knows that either he or C will be executed. Hence, the probability that he will be executed is now only $\frac{1}{2}$. Hence, merely by asking the jailer his question, the prisoner reduced the probability that he would be executed from $\frac{2}{3}$ to $\frac{1}{2}$, because he could go through exactly the same reasoning regardless of which answer the jailer gave. Discuss what is wrong with prisoner A's reasoning. (Do not write more than 10 lines.)

8. A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function $F_Y(y) = P(Y \leq y) = 1 - \frac{c}{y^2}, 1 \leq y \leq \infty$.

   (a) Determine the value of c.

   (b) Find $f_Y(y)$, the pdf of Y.

   (c) If the low-water mark is reset at zero and we use a unit of measurement which is $\frac{1}{10}$ of that given previously, the high-water marks becomes $Z = 10(Y - 1)$. Find $F_Z(z)$.

9. Seven balls are distributed randomly into seven cells. Let $X_i =$ the number of cells containing exactly i balls. What is the probability distribution of X_3? (That is, find $P(X_3 = x)$ for every possible x).

10. Exercise 1.52 (page 44) from C&B.