14.381 STATISTICS
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PROBLEM SET 10
SOLUTIONS

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1. C&B 8.2
Let $X \sim \text{Poisson}(\lambda)$, and we observed $X=10$. To assess if the accident rate has dropped, we could calculate

$$P(X \le 10 \mid \lambda = 15) = \sum_{i=0}^{10} \frac{e^{-15} 15^i}{i!} = e^{-15}\left[1 + 15 + \frac{15^2}{2!} + \dots + \frac{15^{10}}{10!}\right]$$

$$\approx .11846$$

This is not a really low value, so we can accept the hypothesis that $\lambda = 15$.

2. The LRT statistic is

$$\lambda(y) = LRT(y) = \frac{\sup_{\Theta} L(\theta \mid y_1, \dots, y_n)}{\sup_{\theta \le \theta_0} L(\theta \mid y_1, \dots, y_n)}$$

Let $y = \sum_{i=1}^{m} y_i$, and note that the MLE in the denominator is $\min\{y/m, \theta_0\}$ while the numerator has y/m as the MLE. Thus,

$$\lambda(y) = \begin{cases} 1 & \text{if } y/m \le \theta_0 \\ \dfrac{(y/m)^y (1 - y/m)^{m-y}}{(\theta_0)^y (1-\theta_0)^{m-y}} & \text{if } y/m > \theta_0 \end{cases}$$

So we reject if

$$\frac{(y/m)^y (1 - y/m)^{m-y}}{\theta_0^y (1-\theta_0)^{m-y}} > c$$

Then we need to show that this is equivalent to rejecting if $y < b$.
We can do it by showing that $\lambda(y)$ is an increasing function of y:

Take logs and write

$$\log \lambda(y) = y \cdot \log(y/m) + (m-y)\log(1-y/m) - y\log(\theta_0) - (m-y)\log(1-\theta_0).$$

$$\frac{\partial \log \lambda(y)}{\partial y} = \log(y/m) + y\cdot\frac{1}{y/m} - \log(1-y/m) - \frac{m-y}{1-y/m} - \log(\theta_0) + \log(1-\theta_0)$$

$$= \log(y/m) + \cancel{m} - \log\left(\frac{m-y}{m}\right) - \cancel{m} - \log(\theta_0) + \log(1-\theta_0)$$

$$= \log\left(\frac{y/m}{(m-y)/m}\cdot\frac{(1-\theta_0)}{\theta_0}\right) = \log\left(\frac{y/m}{\theta_0}\cdot\frac{(1-\theta_0)}{(m-y)/m}\right)$$

Now, we are looking only at the case when $y/m > \theta_0$, so the number inside of the log is >1, and $\frac{\partial \log \lambda(y)}{\partial y}$ is positive.

Note: I did the proof using Michael's definition for the LRT; if you follow the book's definition you just need to use the LRT as the inverse of the one defined here.

3. C&B 8.13

$X_1, X_2 \sim U(\theta, \theta+1)$

$H_0: \theta = 0$ vs. $H_1: \theta > 0$

Test 1: $\phi_1(X_1)$: Reject H_0 if $X_1 > 0.95$
Test 2: $\phi_2(X_1, X_2)$: Reject H_0 if $X_1 + X_2 > C$

(a) Under H_0, $X_1\,[0, 1]$ and $\alpha_1 = P(X_1 > 0.95 \mid \theta = 0) = 0.05$
We wish to set the size α_2 of $\phi(X_1, X_2)$ equal to 0.05.

$\alpha_2 = P(X_1 + X_2 > C \mid \theta = 0)$

Recall some probability theory:
If $Z = X_1 + X_2$

$$f_Z(z) = \begin{cases} z & 0 \le z \le 1 \\ 2-z & 1 \le z \le 2 \end{cases}$$

If $1 \le C \le 2$, then

$$P(Z > C) = \int_c^2 (2-z)\,dz = \tfrac{1}{2}(2-c)^2$$

$$\Rightarrow \tfrac{1}{2}(2-c)^2 = 0.05 \Rightarrow (2-c)^2 = 0.10$$

$$2 - C = \sqrt{0.10} \Rightarrow C = 2 - \sqrt{.1} \approx 1.68$$

$$\boxed{C = 1.68}$$

(b) Test 1: $P_\theta(X_1 > 0.95) = 0$ if $\theta \le -.05$
$= 1$ if $\theta > 0.95$
$= \theta + 0.5$ if $-.05 < \theta \le 0.95$

Test 2: Let $Z = X_1 + X_2$, $X_1, X_2 \sim U(\theta, \theta+1)$

$$f_Z(z) = \int f_{X_1,X_2}(x_1, z-x_1)\, dx_1$$

$$f_{X_1,X_2}(x_1, z-x_1) = 1 \quad \text{if} \quad \theta \le x_1 \le \theta+1$$
$$\text{and} \quad \theta \le z - x_1 \le \theta+1$$
$$\Rightarrow x_1 \ge z-\theta-1, \quad x_1 \le z-\theta$$

For $2\theta \le z \le 1+2\theta$

$$f_Z(z) = \int_0^{z-\theta} dx_1 = z-\theta-\theta = z-2\theta$$

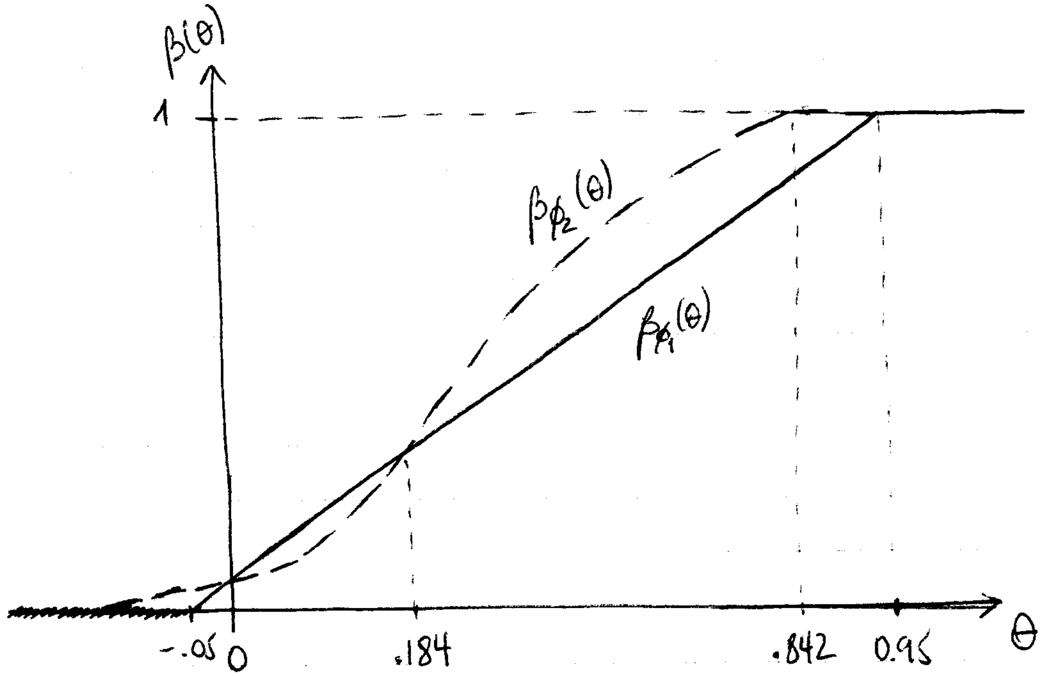
For $1+2\theta \le z \le 2+2\theta$

$$f_Z(z) = \int_{z-\theta-1}^{1+\theta} dx_1 = 1+\theta-z+\theta+1 = 2\theta+2-z$$

$$f_Z(z) = \begin{cases} z-2\theta & 2\theta \le z < 2\theta+1 \\ 2\theta+2-z & 2\theta+1 \le z \le 2\theta+2 \\ 0 & \text{otherwise} \end{cases}$$

Now we can obtain the power function for test 2.

$$P_\theta(Z > C) = \begin{cases} 0 & \text{if } \theta \le (c/2)-1 \\ (2\theta+2-c)^2/2 & \text{if } (c/2)-1 < \theta \le (c-1)/2 \\ 1-(c-2\theta)^2/2 & \text{if } (c-1)/2 < \theta \le c/2 \\ 1 & \text{if } c/2 < \theta \end{cases}$$



So for some θ we have that ϕ_1 is more powerful, but over other interval, ϕ_2 is more powerful.

4. C&B 8.23

(a) The test is Reject H_0 if $X > 1/2$. So the power function is

$$\beta(\theta) = P_\theta(X > 1/2) = \int_{1/2}^{1} \frac{\Gamma(\theta+1)}{\Gamma(\theta)\Gamma(1)} x^{\theta-1}(1-x)^{1-1}\,dx$$

$$= \theta \frac{1}{\theta} x^{\theta}\Big|_{1/2}^{1} = 1 - \frac{1}{2^\theta}$$

The size is $\sup_{\theta \in H_0} \beta(\theta) = \sup_{\theta \le 1} \left(1 - 1/2^\theta\right) = 1 - 1/2 = 1/2.$

(b) By the Neyman-Pearson Lemma, the most powerful test of $H_0: \theta = 1$ vs. $H_1: \theta = 2$ is given by Reject H_0 if $f(x|2)/f(x|1) > k$ for some $k \ge 0$. Substitute the beta pdf to obtain

$$\frac{f(x|2)}{f(x|1)} = \frac{\frac{1}{\beta(2,1)} x^{2-1}(1-x)^{1-1}}{\frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}} = \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} x = 2x$$

Thus, the MP test is reject H_0 if $X > k/2$. We now use the level α to determine k. We have

$$\alpha = \sup_{\theta \in \Theta_0} \beta(\theta) = \beta(1) = \int_{k/2}^{1} f_X(x|1)\,dx = \int_{k/2}^{1} \frac{1}{\beta(1,1)} x^{1-1}(1-x)^{1-1}\,dx$$

$$= 1 - \frac{k}{2}$$

Thus, $1 - k/2 = \alpha$, so the most powerful α level test is reject H_0 if $X > 1-\alpha$.

(c) For $\theta_2 > \theta_1$, $f(x|\theta_2)/f(x|\theta_1) = (\theta_2/\theta_1)\, x_2^{\theta_2-\theta_1}$, an increasing function of x_2 because $\theta_2 > \theta_1$. So this family has MLR. By the Karlin-Rubin Theorem, the test that rejects H_0 if $X > t$ is the UMP test of its size. By the argument in part (b), use $t = 1-\alpha$ to get size α.