14.381 Solutions Problem Set 1
Statistics Fall, 2004

TA: José Tessada
(tessada@mit.edu)

1. (a) sample space: $\Omega = \{t : t \geqq 0\}$ if we assume that fractions of an hour are recorded.
event space: we can use the standard σ-algebra that we use for the real line (see example 1.2.3 in C&B)
probability measure: in general one would expect probability of failure not to be constant for all t; problems like this are generally modeled using some exponential functions. We often use the exponential distribution for situations like this

$$pdf: \qquad f_x(x) = \frac{1}{\beta}\exp(-x/\beta), \qquad 0 \leqq x < \infty, \quad \beta > 0$$

(b) sample space: $\Omega = \{0, 1, 2\}$
event space: $\mathcal{A} = \{\phi, \Omega, \{0\}, \{1\}\, \{2\}, \{0, 1\}, \{0, 2\}, \{1, 2\}\}$
probability measure: $P(\phi) = 0$
$P(\Omega) = 1$
$P(\{0\}) = \frac{1}{4}$
$P(\{1\}) = \frac{1}{2}$
$P(\{2\}) = \frac{1}{4}$
$P(\{0, 1\}) = \frac{3}{4}$
$P(\{0, 2\}) = \frac{1}{2}$
$P(\{1, 2\}) = \frac{3}{4}$

(c) sample space: Ω = {student does micro homework, student does stats homework}
event space: $\mathcal{A}$ = {ϕ, Ω, {student does micro homework}, {student does stats homework}}
probability measure: $P(\phi) = 0,\ P(\Omega) = 1,\ P(\text{micro}) = \frac{3}{4},\ P(\text{stats}) = \frac{1}{4}$.

2. (a) There are $\binom{7}{4} = 35$ ways to choose the academics, and $\binom{5}{2} = 10$ ways to choose the industrialists, hence there are 350 ways to form the committee.

(b) Ways to choose 3 academic women: $\binom{4}{1}\binom{3}{3} = 4$
Ways to choose 2 academic women: $\binom{4}{2}\binom{3}{2} = 18$
Ways to choose 1 industrialist woman: $\binom{2}{1}\binom{3}{1} = 6$
Ways to choose 2 industrialist women: $\binom{2}{2}\binom{3}{0} = 1$
3 academic women and 1 industrialist woman: $4 \cdot 6 = 24$
3 academic women and 2 industrialist women: $4 \cdot 1 = 4$
2 academic women and 2 industrialist women: $18 \cdot 1 = 18$
Thus there are a total of $24 + 4 + 18 = 46$ ways to form the committee with at least four women.

(c) Using Bayes' rule,
$P(\text{Susan} = \text{Academic} \mid \text{Susan chosen}) =$

$$\frac{P(\text{chosen} \mid \text{Academic})\, P(\text{Academic})}{P(\text{chosen} \mid \text{Academic})\, P(\text{Academic}) + P(\text{chosen} \mid \text{Industrialist})\, P(\text{Industrialist})}.$$

$P(\text{Susan chosen} \mid \text{Susan} = \text{Academic}) = \frac{4}{7}$.
$P(\text{Susan chosen} \mid \text{Susan} = \text{Industrialist}) = \frac{2}{5}$.
$P(\text{Susan} = \text{Academic}) = \frac{3}{5}$.
$P(\text{Susan} = \text{Industrialist}) = \frac{2}{5}$.
Hence

$$P(\text{Susan} = \text{Academic} \mid \text{Susan chosen}) = \frac{\frac{4}{7} \cdot \frac{3}{5}}{\frac{4}{7} \cdot \frac{3}{5} + \frac{2}{5} \cdot \frac{2}{5}} = \frac{15}{22}.$$

(d) We proceed as in part (c), but now there are four scenarios to consider.
Let $\Omega = \{(A,A),(A,I),(I,A),(I,I)\}$ denote the set of possible professions of Susan and Peter, respectively.
Then

$$P\left((I,I) \mid \text{both chosen}\right) = \frac{P\left(\text{both chosen} \mid (I,I)\right) P\left(I,I\right)}{\sum_{\omega\in\Omega} P\left(\text{both chosen} \mid \omega\right) P\left(\omega\right)}.$$

$$\begin{aligned}
P\left(\text{both chosen} \mid (I,I)\right) &= \frac{1}{\binom{5}{2}} = \frac{1}{10}.\\
P\left(\text{both chosen} \mid (A,A)\right) &= \frac{\binom{5}{2}}{\binom{7}{4}} = \frac{2}{7}.\\
P\left(\text{both chosen} \mid (A,I)\right) &= \frac{4}{7}\cdot\frac{2}{5} = \frac{8}{35}.\\
P\left(\text{both chosen} \mid (I,A)\right) &= \frac{2}{5}\cdot\frac{4}{7} = \frac{8}{35}.
\end{aligned}$$

$P\left(\text{Susan} = I\right) = \frac{2}{5}$, $P\left(\text{Susan} = A\right) = \frac{3}{5}$, $P\left(\text{Peter} = I\right) = \frac{3}{7}$, $P\left(\text{Peter} = A\right) = \frac{4}{7}$, so $P\left(I,I\right) = \frac{6}{35}$, $P\left(A,A\right) = \frac{12}{35}$, $P\left(A,I\right) = \frac{9}{35}$, $P\left(I,A\right) = \frac{8}{35}$.
Hence

$$P\left((I,I) \mid \text{both chosen}\right) = \frac{\frac{1}{10}\cdot\frac{6}{35}}{\frac{1}{10}\cdot\frac{6}{35} + \frac{2}{7}\cdot\frac{12}{35} + \frac{8}{35}\cdot\frac{9}{35} + \frac{8}{35}\cdot\frac{8}{35}} = \frac{21}{277}.$$

3.

$$\begin{aligned}
B \subset A \quad &\Rightarrow \quad B \cap A = B\\
P(C \mid B) = P(C \mid B) \quad &\Leftrightarrow \quad \frac{P(C\cap A)}{P(A)} = \frac{P(C\cap B)}{P(B)}\\
&\Leftrightarrow \quad \frac{P(C\cap A)}{P(A)} = \frac{P(C\cap (B\cap A))}{P(B\cap A)}\\
&\Leftrightarrow \quad \frac{P(B\cap A)}{P(A)} = \frac{P((C\cap A)\cap B)}{P(C\cap A)}\\
&\Leftrightarrow \quad P(B \mid A) = P(B \mid A\cap C)
\end{aligned}$$

4. $P(A\cup B) = P(A) + P(B) - P(A\cap B) = \frac{3}{8} + \frac{3}{8} - P(A\cap B)$,
hence $\frac{3}{4} = \frac{3}{8} + \frac{3}{8} - P(A\cap B) \Rightarrow P(A\cap B) = 0$.
Likewise, $P(A\cap C) = 0$ and $P(B\cap C) = 0$

So A, B, and C are disjoint events.
Therefore

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) = \frac{9}{8} > 1,$$

which is not possible

5. $P(A) = \frac{1}{3}$ and $P(B^C) = \frac{1}{4} \Rightarrow P(B) = 1 - \frac{1}{4} = \frac{3}{4}$
If A and B disjoint, then $P(A\cup B) = P(A) + P(B) = \frac{1}{3} + \frac{3}{4} = \frac{13}{12} > 1$, which is not possible.
Therefore A and B cannot be disjoint.

If A and B are independent, then $P(A \mid B) = P(A)$, *i.e.*, $P(A\cap B) = P(A)P(B)$ and $P(B \mid A) = P(B)$ so $P(A\cap B) = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$, which is between 0 and 1. A and B can be independent.

Example: We roll one die, and flip 2 coins.

$P(A) = P(\text{die} \in \{1,2\}) = \frac{1}{3}$

$P(B) = P(\text{coins} \in \{HH, HT, TH\}) = \frac{3}{4}$

$P(A\cap B) = \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$

6. Let h_1 and h_2 denote the number of heads on day 1 and day 2, respectively.

   (a) $P\left(\text{Mary washed dishes both days}\right) = P\left(h_1 \leq 2\right) \cdot P\left(h_2 \leq 2\right) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$.

   (b) $P\left(\text{John washed dishes on day 2} \mid \text{no one washed dishes on day 1}\right) = P\left(h_2 \geq 3\right) = \frac{1}{2}$.

   (c) $P\left(\text{Mary washed dishes on day 2} \mid \text{no one washed dishes on day 1}\right) = P\left(h_2 \leq 2\right) = \frac{1}{2}$.

   (d) $P\left(\text{No one washed dishes on day 1} \mid \text{John washed dishes on day 2}\right) =$

$$\frac{P\left(\text{No one washed dishes on day 1 and John washed dishes on day 2}\right)}{P\left(\text{John washed dishes on day 2}\right)}.$$

   $P\left(\text{No one washed dishes on day 1 and John washed dishes on day 2}\right) = P\left(h_1 = 3\right) \cdot P\left(h_2 \geq 3\right) = \frac{5}{16} \cdot \frac{1}{2} = \frac{5}{32}$.
   $P\left(\text{John washed dishes on day 2}\right) = P\left(h_1 = 3\right) \cdot P\left(h_2 \geq 3\right) + P\left(h_1 \neq 3\right) \cdot P\left(h_2 \geq 4\right) = \frac{5}{16} \cdot \frac{1}{2} + \frac{11}{16} \cdot \frac{3}{16} = \frac{73}{256}$.
   Thus $P\left(\text{No one washed dishes on day 1} \mid \text{John washed dishes on day 2}\right) = \frac{\frac{5}{32}}{\frac{73}{256}} = \frac{40}{73}$.

7. See pages 21-23 in C&B (yes! the solution is in the book!)

8. $F_y(y) = P(Y \leq y) = 1 - \frac{c}{y^2}, \quad 1 \leq y \leq \infty$

   (a) $F_y(\infty) = 1$
   $F_y(1) = 0 \Leftrightarrow 1 - \frac{c}{1} = 0 \Leftrightarrow c = 1$

   (b) $f_y(y) = \frac{\partial}{\partial y} F_y(y) = \frac{2c}{y^3} = \frac{2}{y^3} \quad \text{for } 1 \leq y \leq \infty$
   $f_y(y) = 0 \quad \text{for } y < 1$

   (c) $z = 10(y-1)$
   $F_z(z) = P(Z \leq z) = P(10(y-1) \leq z) = P(Y - 1 \leq \frac{z}{10}) = P(Y \leq \frac{z}{10} + 1)$
   $= f_y\left(\frac{z}{10} + 1\right) = 1 - \frac{1}{\left(\frac{z}{10}+1\right)^2}$
   $0 \leq z \leq \infty$
   $F_z(z) = 0 \quad \text{for } z < 0$

9. $P(x_3 = 2) = P(\text{2 cells with 3 balls and 1 cell with 1 ball}) \qquad (3,3,1)$

$$\underbrace{\binom{7}{2}}_{\substack{\text{location of}\\ \text{the 2 cells w/}\\ \text{3 balls}}} \quad \underbrace{\binom{5}{1}}_{\substack{\text{location of}\\ \text{the 1 cell w/}\\ \text{1 ball}}} \quad \underbrace{\binom{7}{3}}_{\substack{\text{3 balls out of 7}\\ \text{for the 1st cell w/}\\ \text{2 balls}}} \quad \underbrace{\binom{4}{3}}_{\substack{\text{3 balls out of}\\ \text{the 4 balls left}\\ \text{to put in the 2nd}\\ \text{cell w/ 2 balls}}} = \frac{7\times 6}{2} \times 5 \times \frac{7\times6\times5}{2\times3} \times 4 = 14700$$

$$P(x_3 = 2) = \frac{14700}{7^7} = 0.018$$
$$(7^7 = \#\text{ of possible arrangements for the 7 balls in 7 cells, all equally likely})$$

$P(x_3 = 1)$: 1 cell with 3 balls and 2 cells with 2 balls (3,2,2)
or 1 cell with 3 balls and 1 cell with 4 balls (4,3)
or 1 cell with 3 balls and 4 cells with 1 ball (3,1,1,1)
or 1 cell with 3 balls and 1 cell with 2 and 2 cells with 1 ball (3,2,1,1)
$x_3 = 1$ and $x_2 = 2$

$x_3 = 1$ and $x_2 = 2$: (3,2,2)

$$\underbrace{\binom{7}{1}}_{\substack{\text{1 cell} \\ \text{for 3 balls}}} \underbrace{\binom{7}{3}}_{\substack{\text{3 balls} \\ \text{for 1st cell}}} \underbrace{\binom{6}{2}}_{\substack{\text{2 cells} \\ \text{for 3 balls}}} \underbrace{\binom{4}{2}}_{\substack{\text{4 balls} \\ \text{for 2 cells} \\ \text{of 2 balls}}} = 7 \times \frac{7!}{3!4!} \times \frac{6!}{2!4!} \times \frac{4!}{2!2!} = 7 \times \frac{7 \times 6 \times 5}{3 \times 2} \times \frac{6 \times 5}{2} \frac{4 \times 3 \times 2}{2 \times 2} = 22050$$

$x_3 = 1$ and $x_4 = 1$: (4,3)

$$\binom{7}{1}\binom{7}{3}\binom{6}{1}\binom{4}{4} = 7 \times \frac{7!}{3!4!} \times 6 = 1470$$

$x_3 = 1$ and $x_1 = 4$: (3,1,1,1)

$$\binom{7}{1}\binom{7}{3}\binom{6}{4}\underbrace{4!}_{\substack{\text{ways to allocate} \\ \text{4 balls into 4 cells}}} = 7 \times 35 \times \frac{6 \times 5}{2} \times 4 \times 3 \times 2 = 88200$$

$x_3 = 1$ and $x_2 = 1$ and $x_1 = 2$: (3,2,1,1)

$$\binom{7}{1}\binom{7}{3}\binom{6}{1}\binom{4}{2}\binom{5}{2} 2 = 7 \times 35 \times 6 \times 6 \times \frac{5 \times 4}{2} \times 2 = 176400$$

$P(x_3 = 1) = \frac{22050+1470+88200+176400}{7^7} = 0.35$

$P(x_3 = 0) = 1 - P(x_3 = 1) - P(x_3 = 2) = 1 - 0.35 - 0.018 = 0.632$

10.

$$g(x) = \begin{cases} f(x)/[1 - F(x_0)] & \text{for } x \geq x_0 \\ 0 & \text{for } x < x_0 \end{cases}$$

We apply C&B Theorem 1.6.5:

(a) $\forall x$, $g(x) \geq 0$, since $f(x) \geq 0$ and $F(x_0) < 1$

(b)

$$\int_{-\infty}^{+\infty} g(x)dx = \frac{1}{1 - F(x_0)} \left[\int_{x_0}^{\infty} f(x)dx \right] = \frac{1 - F(x_0)}{1 - F(x_0)} = 1$$