14.381 PS #2 Solutions

Fall 2004 TA: José Tessada
(Thanks to Ziad Nejmeldeen and Josh Fischman)

1. Notice that $0 \leq y \leq 1$.
We have:

$$F_Y(y) = P(Y \leq y) = P(F_X(x) \leq y)$$
$$\overset{ⓐ}{=} P(X \leq F_X^{-1}(y))$$
$$\overset{\text{by definition}}{=} F_X(F_X^{-1}(y)) \quad [= F_X \circ F_X^{-1}(y)]$$

We can easily compute $F_Y(\cdot)$ for $y \in [0,1]$ by noting that

$$P(Y \leq 0) = P(F_X(X) \leq 0) = 0$$
and $$P(Y \leq 1) = P(F_X(X) \leq 1) = 1.$$

Putting the pieces together we obtain:

$$F_Y(y) = \begin{cases} 0 & \text{for } y < 0 \\ y & \text{for } 0 \leq y \leq 1 \\ 1 & \text{for } y > 1 \end{cases}$$

And we can easily show that if $Z \sim U(0,1)$, then

$$F_Z(z) = \begin{cases} 0 & \text{for } z < 0 \\ z & \text{for } 0 \leq z \leq 1 \\ 1 & \text{for } z > 1 \end{cases}$$

$\Rightarrow$ $Y = F_X(x)$ is uniformly distributed between 0 and 1.

ⓐ here we use the fact that the cdf of X is strictly increasing on x.

2. C&B 2.4

$$f(x) = \begin{cases} (1/2)\,\lambda e^{-\lambda x} & x \geq 0 \\ (1/2)\cdot \lambda e^{\lambda x} & x < 0 \end{cases} \qquad \lambda > 0$$

(a) · $f(x) \geq 0 \quad \forall x$

· $$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_{0}^{\infty} \frac{1}{2}\lambda e^{-\lambda x}\,dx$$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

$\Rightarrow f(x)$ is a pdf.

(b) We can write:

$$P(X \leq t) = \begin{cases} \int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\,dx & t < 0 \\ \underbrace{\int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx}_{=1/2} + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx & t \geq 0 \end{cases}$$

$$\int_{-\infty}^{t} \frac{1}{2}\lambda e^{\lambda x}\,dx = \frac{1}{2} e^{\lambda x}\Big|_{\infty}^{t} = \frac{1}{2} e^{\lambda t}$$

$$\int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx = -\frac{1}{2} e^{-\lambda x}\Big|_{0}^{t} = -\frac{1}{2} e^{-\lambda t} + \frac{1}{2}$$

$$P(X \leq t) = \begin{cases} \frac{1}{2} e^{\lambda t} & t < 0 \\ 1 - \frac{1}{2} e^{-\lambda t} & t \geq 0 \end{cases}$$

(c) by definition of $|\cdot|$, $P(|X| < t) = 0$

$$P(|X| \leq t) = P(-t \leq X \leq t)$$

$$= \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx = \frac{1}{2}\left(1 - e^{-\lambda t}\right) + \frac{1}{2}\left(-e^{-\lambda t} + 1\right)$$

$$P(|X| \le t) = 1 - e^{-\lambda t} \quad //$$

Notice that you can also write:

$$P(-t \le X \le t) = P(X \le t) - P(X \le -t)$$

this is easy to show; using the results from (b):

$$P(-t \le X \le t) = \int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx + \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx - \int_{-\infty}^{-t} \frac{1}{2}\lambda e^{\lambda x}\,dx$$

$$= \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx + \left[\int_{-\infty}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx - \int_{-\infty}^{-t} \frac{1}{2}\lambda e^{\lambda x}\,dx\right]$$

$$= \int_{0}^{t} \frac{1}{2}\lambda e^{-\lambda x}\,dx + \int_{-t}^{0} \frac{1}{2}\lambda e^{\lambda x}\,dx \quad //$$

3. C&B 2.13

$$P(X=1) = P(HT \text{ or } TH) = p(1-p) + (1-p)p$$

$$P(X=2) = P(HHT \text{ or } TTH) = p^2(1-p) + (1-p)^2 p$$

$$\vdots$$

$$P(X=x) = \begin{cases} p^x(1-p) + (1-p)^x p & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \sum_{x=1}^{\infty} x\left[(1-p)^x p + p^x(1-p)\right] = \sum_{x=1}^{\infty} x(1-p)^x p + \sum_{x=1}^{\infty} x p^x (1-p)$$

Now, a trick...

$$\sum_{x=1}^{\infty} x p^{x-1} = \frac{\partial\left(\sum_{x=1}^{\infty} p^x\right)}{\partial p} = \frac{\partial\left(\frac{p}{1-p}\right)}{\partial p} = \frac{1}{(1-p)^2}$$

Plugging in this formula we obtain

$$E(X) = p(1-p)\left[\sum_{x=1}^{\infty} x(1-p)^{x-1} + \sum_{x=1}^{\infty} x p^{x-1}\right] = p(1-p)\left[\frac{1}{p^2} + \frac{1}{(1-p)^2}\right]$$

$$E(x) = \frac{1-p}{p} + \frac{p}{1-p} = \frac{(1-p)^2 + p^2}{p(1-p)}$$

Note that as $p \to 0$ or $p \to 1$, $E(x) \to \infty$.

4. C&B 2.23

(a) Partition the interval $(-1,1)$ into $A_0 = \{0\}$, $A_1 = (-1,0)$ and $A_2 = (0,1)$, so that $P(X \in A_0) = 0$.
We can now define the functions $g_1(x) = x^2$ - monotone on A_1 - and $g_2(x) = x^2$ - monotone on A_2.
The inverse functions are $g_1^{-1}(y) = -\sqrt{y}$ ∧ $g_2^{-1}(y) = \sqrt{y}$

Also, $$\left|\frac{d}{dy} g_1^{-1}(y)\right| = \left|\frac{d}{dy} g_2^{-1}(y)\right| = \frac{1}{2\sqrt{y}}$$

Let $\mathcal{Y} = \{y : y = g_i(x) \text{ for some } x \in A_i\} = (0,1)$.
We can apply Theorem 2.1.8 from the book:

$$f_Y(y) = \begin{cases} \sum_{i=1}^{2} f_X(g_i^{-1}(y)) \left|\frac{d}{dy} g_i^{-1}(y)\right| & \text{for } y \in \mathcal{Y} \\ 0 & \text{otherwise} \end{cases}$$

$$= \frac{1}{2}(1-\sqrt{y})\frac{1}{2\sqrt{y}} + \frac{1}{2}(1+\sqrt{y})\frac{1}{2\sqrt{y}}, \quad \text{for } y \in (0,1)$$

$$= \frac{1}{2\sqrt{y}} \quad \text{for } y \in (0,1)$$

An alternative way to do it: use the same procedure used in question 2, part (c).

(b) $E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 \frac{1}{2} y^{1/2}\, dy = \frac{1}{3} y^{3/2} \Big|_0^1 = \frac{1}{3}$

$E(Y^2) = \int_0^1 y^2 f_Y(y)\, dy = \int_0^1 \frac{1}{2} y^{3/2}\, dy = \frac{1}{5} y^{5/2} \Big|_0^1 = \frac{1}{5}$

$Var(Y) = E(Y^2) - E(Y)^2 = \frac{1}{5} - \frac{1}{9} = \frac{4}{45}$ //

5. C&B 2.25

(a) $X, -X$ are identically distributed if $F_X(t) = F_{-X}(t)$

Note: you can also define $Y = -X$ and show that $f_Y(y) = f_X(y)$

$F_X(t) = P(X \le t) = \int_{-\infty}^{t} f_X(x)\, dx = \int_{-t}^{\infty} f_X(x)\, dx$

$f_X(\cdot)$ is even!

$= P(X \ge -t) = P(-X \le t) = F_{-X}(t)$

(b) $M_X(t)$ is symmetric about 0 $\Leftrightarrow M_X(t) = M_X(-t)$

$M_X(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{xt} f_X(x) dx$

$M_X(-t) = E(e^{-xt}) = \int_{-\infty}^{\infty} e^{-xt} f_X(x)\, dx = \int_{-\infty}^{\infty} e^{-xt} f_X(-x)\, dx$

even function

define $Z = -X$, then $dZ = -dX$

$M_X(-t) = \int_{-\infty}^{\infty} e^{zt} f_X(z)\, dz = \int_{-\infty}^{\infty} e^{xt} f_X(x)\, dx = M_X(t)$ //

where we made use of the fact that $\int_{\infty}^{-\infty} f(x) dx = -\int_{-\infty}^{\infty} f(x) dx$.

Another proof:

$M_X(t) = \int_{-\infty}^{0} e^{tx} f_X(x)\, dx + \int_0^{\infty} e^{tx} f_X(x) dx = \int_0^{\infty} e^{t(-x)} f_X(-x)\, dx + \int_{-\infty}^{0} e^{t(-x)} f_X(-x)\, dx$

$= \int_{-\infty}^{\infty} e^{-tx} f_X(-x)\, dx = \int_{-\infty}^{\infty} e^{-tx} f_X(x)\, dx = M_X(-t)$ //

6. C&B 2.30

(a) $f(x) = \frac{1}{c} \qquad 0 < x < c$

$$M_X(t) = E(e^{xt}) = \int_0^c e^{xt} \cdot 1/c \, dx$$
$$= \frac{1}{c} \left[\frac{1}{t} e^{xt} \right] \Big|_0^c$$
$$= \frac{e^{ct} - 1}{ct} \quad //$$

(b) $f(x) = \frac{2x}{c^2} \qquad 0 < x < c$

$$M_X(t) = \frac{2}{c^2} \int_0^c e^{xt} x \, dx$$

(integrating by parts)
$$= \frac{2}{c^2} \left(\frac{x e^{xt}}{t} \right) \Big|_0^c - \frac{2}{c^2} \int_0^c \frac{e^{xt}}{t} dx$$
$$= \frac{2}{c^2} \left[\frac{c e^{ct}}{t} - \frac{1}{t^2} e^{xt} \Big|_0^c \right]$$
$$= \frac{2}{c^2} \left[\frac{c e^{ct}}{t} - \frac{1}{t^2} e^{ct} + \frac{1}{t^2} \right]$$
$$M_X(t) = \frac{2 e^{ct}}{ct} - \frac{2 e^{ct}}{c^2 t^2} + \frac{2}{c^2 t^2} \quad //$$

7. C&B 2.33 (a),

(a)

by definition of e

$$M_X(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{(e^t \lambda)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$$
$$= e^{\lambda(e^t - 1)}$$

$$E(X) = \frac{d}{dt} M_X(t) \Big|_{t=0} = e^{\lambda(e^t-1)} \lambda e^t \Big|_{t=0} = \lambda \quad //$$

$$E(X^2) = \frac{d^2}{dt^2} M_X(t) \Big|_{t=0} = \lambda e^t e^{\lambda(e^t-1)} \lambda e^t + \lambda e^t e^{\lambda(e^t-1)} \Big|_{t=0}$$
$$= \lambda^2 + \lambda$$

$$Var(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

(c) the trick here is to complete the squares:

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x^2-2\mu x-2\sigma^2 tx+\mu^2)/2\sigma^2} dx$$

Now we complete the square in the exponential

$$x^2 - 2\mu x - 2\sigma^2 tx + \mu^2 = x^2 - 2(\mu+\sigma^2 t)x + \mu^2$$

$$= [x-(\mu+\sigma^2 t)]^2 - (\mu+\sigma^2 t)^2 + \mu^2 = [x-(\mu+\sigma^2 t)]^2 - (2\mu\sigma^2 t + (\sigma^2 t)^2)$$

$$M_x(t) = e^{[2\mu\sigma^2 t + (\sigma^2 t)^2]/2\sigma^2} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}(x-(\mu+\sigma^2 t))^2} dx}_{\text{this is a pdf !!}}$$

$\Rightarrow = 1.$

$$M_x(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$$

$$E(X) = \frac{d}{dt} M_x(t)\Big|_{t=0} = (\mu+\sigma^2 t)e^{\mu t+\sigma^2 t^2/2}\Big|_{t=0} = \mu$$

$$E(X^2) = \frac{d^2}{dt^2} M_x(t)\Big|_{t=0} = (\mu+\sigma^2 t)^2 e^{\mu t+\sigma^2 t^2/2} + \sigma^2 e^{\mu t+\sigma^2 t/}$$

NOT DUE

8. $M_Y(t) = e^{c[M_x(t)-1]}$ $\qquad M_Y^{(i)}(t) = \frac{d^i}{dt^i} M_Y(t)$

$$M_Y^{(1)}(t) = c M_x^{(1)}(t) e^{c[M_x(t)-1]}$$

$$M_Y^{(2)}(t) = c M_x^{(2)}(t) e^{c[M_x(t)-1]} + c^2 M_x^{(1)}(t)^2 e^{c[M_x(t)-1]}$$

$$E(Y) = M_Y^{(1)}(t)\Big|_{t=0} = c M_x^{(1)}(t)\Big|_{t=0} e^{c[1-1]} = c \cdot E(X) = c\mu$$

$$E(Y^2) = M_Y^{(2)}(t)\Big|_{t=0} = c E(X^2) + c^2 E(X)^2 = c(\sigma^2+\mu^2) + c^2\mu^2$$

where $\sigma^2 = Var(X)$

$$Var(Y) = c(\sigma^2+\mu^2) + c^2\mu^2 - (c\mu)^2 = \boxed{c(\sigma^2+\mu^2) = Var(Y)}$$

9. C&B 2.34

From lecture we know that

$$M_X(t) = e^{t^2/2} \quad \text{if} \quad X \sim N(0,1)$$

Then we can compute the moments using the mgf.

$$\left.\begin{aligned}
M_X^{(1)}(t) &= t\,e^{t^2/2} \\
M_X^{(2)}(t) &= (t^2+1)\,e^{t^2/2} \\
M_X^{(3)}(t) &= (t^3+3t)\,e^{t^2/2} \\
M_X^{(4)}(t) &= (t^4+6t^2+3)\,e^{t^2/2} \\
M_X^{(5)}(t) &= (t^5+10t^3+15t)\,e^{t^2/2}
\end{aligned}\right\} \Rightarrow \left\{\begin{aligned}
E(X) &= 0 \\
E(X^2) &= 1 \\
E(X^3) &= 0 \\
E(X^4) &= 3 \\
E(X^5) &= 0
\end{aligned}\right.$$

$$E(Y) = \sqrt{3}\,\frac{1}{6} - \sqrt{3}\,\frac{1}{6} = 0$$

$$E(Y^2) = 3 \cdot \frac{1}{6} + 3\,\frac{1}{6} = 1$$

$$E(Y^3) = 3\sqrt{3}\,\frac{1}{6} - 3\sqrt{3}\,\frac{1}{6} = 0$$

$$E(Y^4) = 9 \cdot \frac{1}{6} + 9 \cdot \frac{1}{6} = 3$$

$$E(Y^5) = 9\sqrt{3} \cdot \frac{1}{6} - 9\sqrt{3} \cdot \frac{1}{6} = 0 \;//$$