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1. (a) p. 99 C&B
$X \sim \text{gamma}(\alpha,\beta)$

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \qquad 0 < x < \infty,\ \alpha,\beta > 0$$

to check that it is a pdf, use the change of variable $t = \beta x$ and the definition of the gamma function.

(b)
$$E(x) = \int_0^\infty \frac{x \cdot x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^\alpha e^{-x/\beta} dx$$

change of variable $t=\beta x$
$$= \frac{1}{\Gamma(\alpha)\beta^\alpha} \beta^{\alpha+1} \underbrace{\int_0^\infty t^\alpha e^{-t} dt}_{\Gamma(\alpha+1)} \quad (*)$$

by def. of the $\Gamma(\cdot)$ function
$$= \frac{\beta\Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta\,\alpha\,\Gamma(\alpha)}{\Gamma(\alpha)} = \alpha\beta$$

property of $\Gamma(\cdot)$ [C&B. p. 99]

I didn't use the fact that the pdf integrates to 1, but if in (*) instead of using the change of variables you "complete" the integral [which is way trickier because you will still need to change variables] to get the pdf, then you can claim that $\int f_X(x)dx = 1$.

If you want to check that $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ for $\alpha > 0$, start from the def. of $\Gamma(\alpha)$ and integrate by parts the RHS.

$$E(x^2) = \int_0^\infty \frac{x^2 x^{\alpha-1} e^{-x/\beta}}{\Gamma(\alpha)\beta^\alpha} dx = \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{\alpha+1} e^{-x/\beta} dx$$

Using the same trick as in (a) we obtain

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$$E(X^2) = \alpha(\alpha+1)\beta^2$$
because $\Gamma(\alpha+2) = \alpha(\alpha+1)\Gamma(\alpha)$.

$$Var(X) = E(X^2) - E(X)^2 = \alpha(\alpha+1)\beta^2 - \alpha^2\beta^2 = (\alpha^2+\alpha)\beta^2 - \alpha^2\beta^2$$

$$Var(X) = \alpha\beta^2 //$$

(c) $$M_X(t) = E e^{tX} = \int_0^\infty \frac{e^{tx}\, x^{\alpha-1} e^{-x/\beta}}{\beta^\alpha \Gamma(\alpha)}\, dx$$

$$= \int_0^\infty \frac{x^{\alpha-1} e^{(t-1/\beta)x}}{\beta^\alpha\Gamma(\alpha)}\, dx = \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(1/\beta - t)x}\, dx$$

Change variable
$y = (\frac{1}{\beta} - t)x$
$dy = (\frac{1}{\beta} - t)dx$
$x = (\frac{1}{\frac{1}{\beta}-t})y$

$$= \frac{1}{\beta^\alpha\Gamma(\alpha)} \int_0^\infty \left(\frac{1}{\frac{1}{\beta}-t}\right)^{\alpha-1} y^{\alpha-1} e^{-y} \frac{1}{(\frac{1}{\beta}-t)}\, dy$$

transf. valid iff $t < \frac{1}{\beta}$
so $y > 0$ if $x > 0$ /

$$= \frac{1}{\beta^\alpha\Gamma(\alpha)} \frac{1}{(\frac{1}{\beta}-t)^\alpha} \int_0^\infty y^{\alpha-1} e^{-y}\, dy$$

$$= \frac{\cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\, \beta^\alpha (\frac{1}{\beta}-t)^\alpha} = \frac{1}{(1-\beta t)^\alpha} //$$

2. C&B 3.8.

(a) We want $P(X>N) < 0.01$ where X is a random variable (r.v.) with binomial$(1000, 1/2)$.

With the customers being randomly chosen we fix $p = 1/2$.

$\Rightarrow$ we need
$$P(X>N) = \sum_{x=N+1}^{1000} \binom{1000}{x} \left(\frac{1}{2}\right)^x \left(1-\frac{1}{2}\right)^{1000-x} < 0.01$$

$$\Rightarrow \left(\frac{1}{2}\right)^{1000} \sum_{x=N+1}^{1000} \binom{1000}{x} < 0.01$$

N is the smallest integer that satisfies the previous inequality.

Solution: $N = 537$.

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(b) Following the book we take approximation using $X \sim N(500, 250) \Rightarrow \mu = 1000 \cdot \frac{1}{2}$

$\sigma^2 = 1000 \cdot \frac{1}{2} \cdot \frac{1}{2}$

Then, $P(X > N) = P\left(\frac{X - 500}{\sqrt{250}} > \frac{N - 500}{\sqrt{250}} \right) < 0.01$

thus, $P\left(Z > \frac{N-500}{\sqrt{250}} \right) < 0.01$

where $Z \sim N(0,1)$

We can use the normal standard table (or Excel or Matlab or ...) to get

$P(Z > 2.33) \approx 0.0099 < .01$

$$\Rightarrow \frac{N-500}{\sqrt{250}} = 2.33 \Rightarrow N \approx 537$$

3. C&B 3.9

(a) We can think of each one of the 60 children entering kindergarten as 60 independent Bernoulli trials with probability of success (a twin birth) of approximately 1/90. The probability of having 5 or more successes approximates the probability of having 5 or more sets of twins entering kindergarten. Then $X \sim \text{binomial}(60, 1/90)$ and

$$P(X \geq 5) = 1 - \sum_{x=0}^{4} \binom{60}{x} \left(\frac{1}{90}\right)^x \left(1 - \frac{1}{90}\right)^{60-x} = 0.0006$$

which is small and may be rare enough to be newsworthy.

(b) Let X be the number of elementary schools in New York that have 5 or more sets of twins entering kindergarten. Then the probability of interest is $P(X \geq 1)$ where $X \sim (310, 0.0006)$.

$\Rightarrow P(X \geq 1) = 1 - P(X=0) = 0.1698$

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(c) Let X be the number of states that have 5 or more sets of twins entering kindergarten during any of the last ten years. Then the probability of interest is $P(X \geq 1)$ where $X \sim \text{binomial}(500, 0.1698)$.

$$\Rightarrow P(X \geq 1) = 1 - P(X=0) = 1 - 3.90 \times 10^{-41} \approx 1$$

5. C&B 3.23

(a)
$$\int_\alpha^\infty x^{-\beta-1}\, dx = -\frac{1}{\beta} x^{-\beta} \Big|_\alpha^\infty = \frac{1}{\beta \alpha^\beta}$$

$$\Rightarrow \int_\alpha^\infty \frac{\beta \alpha^\beta}{x^{\beta+1}}\, dx = 1$$

(b) to make it simpler, let's find a formula for $E[X^n]$ first:

$$E[X^n] = \int_\alpha^\infty x^n \frac{\beta \alpha^\beta}{x^{\beta+1}}\, dx = \beta \alpha^\beta \int_\alpha^\infty x^{n-\beta-1}\, dx$$
$$= \beta \alpha^\beta \left[\frac{1}{n-\beta} x^{n-\beta} \Big|_\alpha^\infty \right]$$

now notice that the integral exists if $\beta > n$ (formal argument at the end of the exercise).

$$E[X^n] = \beta \alpha^\beta \cdot \left[\underbrace{\frac{1}{n-\beta} x^{n-\beta} \Big|_{x=\infty}}_{\to 0 \text{ if } n<\beta} - \frac{1}{n-\beta} x^{n-\beta} \Big|_\alpha \right]$$

$$E[X^n] = \frac{\beta \alpha^n}{\beta - n} > 0 \quad \text{given } \beta > n.$$

Notas / Notes:

$$\Rightarrow \quad E[X] = \frac{\beta\alpha}{\beta-1}$$

$$E[X^2] = \frac{\beta\alpha^2}{\beta-2}$$

$$Var[X] = E[X^2] - E[X]^2$$

$$= \frac{\beta\alpha^2}{\beta-2} - \frac{\beta^2\alpha^2}{(\beta-1)^2} = \frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)}$$

Why do we need $\beta > n$? (It is also the answer for (c) if we set $n=2$).

take $\int_\alpha^m x^n \frac{\beta\alpha^\beta}{x^{\beta+1}}\,dx = \beta\alpha^\beta \int_\alpha^m x^{n-\beta-1}\,dx$

(assuming $n>\beta$) $\geq \beta\alpha^\beta \int_\alpha^m x^{-1}\,dx$

$$= \beta\alpha^\beta \left(\log x \Big|_{x=\alpha}^{m} \right)$$

and $\beta\alpha^\beta \left(\log x \Big|_{x=\alpha}^{m} \right) \longrightarrow \infty$ as $m \longrightarrow \infty$

so the integral doesn't exist (is infinite!).

Set $n=2 \longrightarrow$ apply the same logic and you can prove it (or at least show it if you just evaluate the function).

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6.

(a) $X \sim U(0,1) \rightarrow$ pdf: $f_X(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow \text{cdf: } F_X(x) = \begin{cases} x & x \in [0,1) \\ 0 & x < 0 \\ 1 & x \geq 1. \end{cases}$$

$$\Rightarrow f_X(x \mid x > a) = \begin{cases} 1/(1-a) & \text{for } x \in (a,1) \\ 0 & \text{otherwise} \end{cases}$$

(b) From part (a) we can see that

$(X \mid X > a) \sim U(a, 1)$

[for a more formal proof, compute the cdf and check it is the same]

$$\Rightarrow E(X \mid X > a) = \frac{1+a}{2}$$

$$Var(X \mid X > a) = \frac{(1-a)^2}{12}$$

(c) $\Pr(X < a) = \Pr\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right)$

since $\frac{X-\mu}{\sigma} \sim N(0,1)$

An easy way to proceed is the following:

$$f(x) = \frac{d}{da} \Pr(X < a) \Big|_{a=x} = \frac{d}{da} \Phi\left(\frac{a-\mu}{\sigma}\right) \Big|_{a=x} = \frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right)$$

which is a way to write a pdf of a normal in terms of the pdf of the normal (standard) ϕ.

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$\Rightarrow$ we can write the pdf of the truncated distribution as:

$$f(x|x>a) = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{P_r(X>a)} = \frac{\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)}{1-\Phi\left(\frac{a-\mu}{\sigma}\right)} \quad // \quad = \frac{1}{\sigma}\cdot\lambda(\alpha)$$

(d) $$E[x|x>a] = \int_a^\infty x\cdot f(x|x>a)\,dx$$

$$= \frac{1}{\sigma\left[1-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}\int_a^\infty x\,\phi\left(\frac{x-\mu}{\sigma}\right)dx$$

$$= \frac{1}{\sigma\left[1-\Phi\left(\frac{a-\mu}{\sigma}\right)\right]}\int_\alpha^\infty (\sigma u+\mu)\,\phi(u)\,\sigma\,du$$

change of variable $u = \frac{x-\mu}{\sigma}$ $\left(\Rightarrow \alpha = \frac{a-\mu}{\sigma}\right)$

$$= \frac{1}{\sigma\left[1-\Phi(\alpha)\right]}\left[\sigma\mu\,\Phi(u) - \sigma^2\phi\,u\right]\Big|_\alpha^\infty$$

$$= \frac{\sigma\mu\left[1-\Phi(\alpha)\right]+\sigma^2\phi(\alpha)}{\sigma\left[1-\Phi(\alpha)\right]}$$

$$E[x|x>a] = \mu+\sigma\cdot\lambda(\alpha) \quad //$$

$$E[x^2|x>a] = \int_a^\infty x^2 f(x|x>a)\,dx$$

$$= \frac{1}{\sigma[1-\Phi(\alpha)]}\int_a^\infty x^2\,\phi\left(\frac{x-\mu}{\sigma}\right)dx$$

apply the same change of variables

$$= \frac{1}{\sigma[1-\Phi(\alpha)]}\int_\alpha^\infty (\sigma u+\mu)^2\,\phi(u)\,\sigma\,du$$

$$= \frac{1}{\sigma[1-\Phi(\alpha)]}\int_\alpha^\infty \left(\sigma^3u^2+2\sigma^2\mu u+\sigma\mu^2\right)\phi(u)\,du$$

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And some nice properties...

$$\int u^2 \phi(u)\, du = \Phi(u) - u\phi(u)$$

$$\int u\, \phi(u)\, du = -\phi(u)$$

$$\int \phi(u)\, du = \Phi(u)$$

$\Rightarrow$ applying the three formulas we obtain

$$E[X^2 | X>a] = \frac{1}{\sigma[1-\Phi(\alpha)]} \Big[\sigma^3 (\Phi(u) - u\phi(u)) - 2\sigma^2 \mu \phi(u) + \sigma\mu^2 \Phi(u)\Big]_\alpha^\infty$$

$$= \frac{\sigma(\sigma^2+\mu^2)(1-\Phi(\alpha)) + \sigma^2(\alpha\sigma + 2\mu)\phi(\alpha)}{\sigma[1-\Phi(\alpha)]}$$

$$= \sigma^2 + \mu^2 + \sigma^2 \alpha \cdot \lambda(\alpha) + 2\sigma\mu\, \lambda(\alpha)$$

Hence,

$$Var(X^2 | X>a) = E(X^2 | X>a) - E(X|X>a)^2$$

$$= \sigma^2 + \mu^2 + \sigma^2 \alpha\, \lambda(\alpha) + 2\sigma\mu\, \lambda(\alpha) - [\mu + \sigma\, \lambda(\alpha)]^2$$

$$= \sigma^2 [1 - \lambda(\alpha)(\lambda(\alpha) - \alpha)] \;//$$

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