1. (a) p. 99 C&B

\[ X \sim \text{gamma}(\alpha, \beta) \]

\[ f(x|\alpha, \beta) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} e^{-x/\beta} \quad 0 < x < \infty \quad \alpha, \beta > 0 \]

To check that it is a pdf, use the change of variable \( t = \beta x \) and the definition of the gamma function.

(b) \[ E(x) = \int_0^\infty x \cdot x^{\alpha-1} e^{-x/\beta} \, dx = \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty x^\alpha e^{-x/\beta} \, dx \]

Change of variable \( t = \beta x \):

\[ = \frac{1}{\Gamma(\alpha) \beta^\alpha} \beta^{\alpha+1} \int_0^\infty t^\alpha e^{-t} \, dt \]

\[ = \frac{\beta \Gamma(\alpha+1)}{\Gamma(\alpha)} = \frac{\beta \cdot \alpha \cdot \Gamma(\alpha)}{\Gamma(\alpha)} = \alpha \beta \]

Property of \( \Gamma(\cdot) \) [C&B, p. 99]

I didn't use the fact that the pdf integrates to 1, but if in (a) instead of using the change of variable you "complete" the integral [which is now thicker because you will still need to change variables] to get the pdf, then you can claim that \( \int f(x) \, dx = 1 \).

If you want to check that \( \Gamma(\alpha+1) = \alpha \Gamma(\alpha) \) for \( \alpha > 0 \) start from the def. of \( \Gamma(\cdot) \) and integrate by parts the RHS.

\[ E(x^2) = \int_0^\infty x^2 x^{\alpha-1} e^{-x/\beta} \, dx = \frac{1}{\Gamma(\alpha) \beta^\alpha} \int_0^\infty x^{\alpha+1} e^{-x/\beta} \, dx \]

Using the same trick as in (a), we obtain...
\[ E(x^2) = \alpha(\alpha+1) \beta^2 \]

Because \( \Gamma(\alpha+2) = \alpha(\alpha+1) \Gamma(\alpha) \).

\[ \text{Var}(x) = E(x^2) - E(x)^2 = \alpha(\alpha+1) \beta^2 - \alpha^2 \beta^2 = (\alpha^2 + \alpha) / \beta^2 - \alpha^2 \beta^2 \]

\[ \text{Var}(x) = \alpha / \beta^2 \]

\( c \) \( M_x(t) = E(e^{tx}) = \int_0^\infty e^{tx} x e^{-x/\beta} \frac{dx}{\beta \Gamma(\alpha)} \]

\[ = \int_0^\infty x e^{-x/\beta} dx = \frac{1}{\beta \Gamma(\alpha)} \int_0^\infty y^{\alpha-1} e^{-y/(\beta-t)} \frac{dy}{(\beta-t)} \]

\[ = \frac{\Gamma(\alpha)}{\beta^\alpha \Gamma(\alpha)} = \frac{1}{(1-\beta t)^\alpha} \]

2. CJB 3.8.

(a) We want \( P(X > N) < 0.01 \) where \( X \) is a random variable (r.v.) with binomial \((1000, 1/2)\).

With the customers being randomly chosen we fix \( p = 1/2 \).

\[ \Rightarrow \text{we need } P(X > N) = \sum_{x=N+1}^{1000} \binom{1000}{x} \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{1000-x} < 0.01 \]

\[ \Rightarrow \left( \frac{1}{2} \right)^{1000} \sum_{x=N+1}^{1000} \binom{1000}{x} < 0.01 \]

\( N \) is the smallest integer that satisfies the previous inequality.

Solution: \( N = 537 \).
(b) Following the book we take approximation using 
\[ X \sim N(500, 250) \Rightarrow \mu = 1000 \cdot \frac{1}{2}, \quad \sigma^2 = 1000 \cdot \frac{1}{2} \cdot \frac{1}{2} \]

Then, 
\[ P(X > N) = P \left( \frac{X - 500}{\sqrt{250}} > \frac{N - 500}{\sqrt{250}} \right) \leq 0.01 \]

thus, 
\[ P \left( \frac{Z}{\sqrt{250}} > 2.33 \right) \approx 0.0099 < 0.01 \]

\[ \Rightarrow \frac{N - 500}{\sqrt{250}} = 2.33 \Rightarrow N \approx 537 \]

3. Ch. 3.9

(a) We can think of each one of the 60 children entering kindergarten as 60 independent Bernoulli trials with probability of success (a twin birth) of approximately 1/90. The probability of having 5 or more successes approximates the probability of having 5 or more sets of twins entering kindergarten. Then 
\[ X \sim \text{binomial} (60, \frac{1}{90}) \]

\[ P(X \geq 5) = 1 - \sum_{x=0}^{4} \binom{60}{x} \left( \frac{1}{90} \right)^x \left( 1 - \frac{1}{90} \right)^{60-x} = 0.0006 \]

which is small and may be rare enough to be newsworthy.

(b) Let \( X \) be the number of elementary schools in New York that have 5 or more sets of twins entering kindergarten. Then the probability of interest is 
\[ P(X \geq 1), \quad \text{where} \quad X \sim (310, 0.0006) \]

\[ \Rightarrow P(X \geq 1) = 1 - P(X=0) = 0.1698 \]
(c) Let \( X \) be the number of states that have 5 or more sets of twins entering kindergarten during any of the last ten years. Then the probability of interest is \( P(X \geq 1) \) where \( X \sim \text{binomial}(500, 0.1698) \).

\[ \Rightarrow P(X \geq 1) = 1 - P(X = 0) = 1 - 3.90 \times 10^{-41} \approx 1 \]

5. (C&B 3.23)

(a) \[
\int_{\alpha}^{\infty} x^{-\beta-1} \, dx = -\frac{1}{\beta} x^{-\beta} \bigg|_{\alpha}^{\infty} = \frac{1}{\beta \alpha^\beta} 
\]

\[ \Rightarrow \int_{\alpha}^{\infty} \frac{\beta \alpha^\beta}{x^{\beta+1}} \, dx = 1 \]

(b) To make it simpler, let's find a formula for \( E[X^n] \) first:

\[
E[X^n] = \int_{\alpha}^{\infty} x^n \frac{\beta \alpha^\beta}{x^{\beta+1}} \, dx = \beta \alpha^\beta \int_{\alpha}^{\infty} x^{n-\beta-1} \, dx 
\]

\[ = \beta \alpha^\beta \left[ \frac{1}{n-\beta} x^{n-\beta} \right]_{\alpha}^{\infty} \]

Now notice that the integral exists if \( \beta > n \) (formal argument at the end of the exercise).

\[
E[X^n] = \beta \alpha^\beta \left[ \frac{1}{n-\beta} x^{n-\beta} \bigg|_{\alpha}^{\infty} - \frac{1}{n-\beta} x^{n-\beta} \bigg|_{\alpha}^{\alpha} \right] 
\]

\[ \to 0 \quad \text{if} \quad n < \beta \]

\[
E[X^n] = \frac{\beta \alpha^n}{\beta - n} > 0 \quad \text{given} \quad \beta > n. 
\]
\[ E[X] = \frac{\beta \alpha}{\beta - 1} \]
\[ E[X^2] = \frac{\beta \alpha^2}{\beta - 2} \]
\[ \text{Var}[X] = E[X^2] - E[X]^2 \]
\[ = \frac{\beta \alpha^2}{\beta - 2} - \left( \frac{\beta \alpha}{\beta - 1} \right)^2 = \frac{\beta \alpha^2}{(\beta - 1)^2(\beta - 2)} \]

Why do we need \( \beta > n \)? (It is also the answer for (c) if we set \( n = 2 \)).

Take \( \int_\alpha^m x^n \frac{\beta \alpha^\beta}{x^{(\beta + 1)}} \, dx = \beta \alpha^\beta \int_\alpha^m x^{n-\beta - 1} \, dx \)
(assuming \( n > \beta \))
\[ = \beta \alpha^\beta \left( \frac{x^n}{\beta} \right) \bigg|_\alpha^m \]
\[ = \beta \alpha^\beta \left( \log x \right) \bigg|_\alpha^m \]
and \( \beta \alpha^\beta (\log x) \bigg|_\alpha^m \rightarrow \infty \) as \( m \rightarrow \infty \)

so the integral doesn't exist. (is infinite!).

Set \( n = 2 \) to apply the same logic and you can prove it (or at least show it, if you just evaluate the function).
6. (a) \( X \sim U(0,1) \rightarrow \text{pdf: } f_X(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

\[ \Rightarrow \text{cdf: } F_X(x) = \begin{cases} x & x \in [0,1) \\ 0 & x \leq 0 \\ 1 & x \geq 1 \end{cases} \]

\[ \Rightarrow f_X(x|X > a) = \begin{cases} \frac{1}{1-a} & x \in (a,1) \\ 0 & \text{otherwise} \end{cases} \]

(b) From part (a) we can see that \( X|X > a \sim U(a,1) \)
[for a more formal proof, compute the cdf and check it is the same]

\[ \Rightarrow E(X|X > a) = \frac{1+a}{2} \]

\[ \text{Var}(X|X > a) = \frac{(1-a)^2}{12} \]

(c) \( Pr(X < a) = Pr\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right) = \Phi\left(\frac{a-\mu}{\sigma}\right) \)

since \( \frac{X-\mu}{\sigma} \sim N(0,1) \)

An easy way to proceed is the following:

\[ f(x) = \frac{d}{da} \text{Pr}(X < a) \bigg|_{a=x} = \frac{d}{da} \Phi\left(\frac{a-\mu}{\sigma}\right) \bigg|_{a=x} = -\frac{1}{\sigma} \phi\left(\frac{x-\mu}{\sigma}\right) \]

which is a way to write a pdf of a normal in terms of the pdf of the normal (standard) \( \phi \).
we can write the pdf of the truncated distribution as

\[ f(x | x > a) = \frac{1}{\sigma} \frac{\phi\left(\frac{x - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{a - \mu}{\sigma}\right)} = \frac{1}{\sigma} \cdot \lambda(x) \]

(d) \[ E[x | x > a] = \int_a^\infty x \cdot f(x | x > a) \, dx \]

\[ = \frac{1}{\sqrt{\pi} \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right]} \int_a^\infty x \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \, dx \]

\[ = \frac{1}{\sqrt{\pi} \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right]} \left[ \sigma \mu \Phi(u) - \sigma^2 \phi(u) \right]_a^\infty \]

\[ = \sigma \mu \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right] + \sigma^2 \phi\left(\frac{a - \mu}{\sigma}\right) \]

\[ E[x | x > a] = \mu + \sigma \cdot \lambda(x) \]

\[ E[x^2 | x > a] = \int_a^\infty x^2 \cdot f(x | x > a) \, dx \]

\[ = \frac{1}{\sqrt{\pi} \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right]} \int_a^\infty x^2 \cdot \phi\left(\frac{x - \mu}{\sigma}\right) \, dx \]

apply the same change of variables

\[ = \frac{1}{\sqrt{\pi} \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right]} \int_a^\infty (\sigma u + \mu)^2 \phi(u) \, \sigma \, du \]

\[ = \frac{1}{\sqrt{\pi} \left[ 1 - \Phi\left(\frac{a - \mu}{\sigma}\right) \right]} \int_a^\infty (\sigma^3 u^2 + 2 \sigma^2 \mu u + \sigma^2 \mu^2) \phi(u) \, du \]
And some nice properties...
\[ \int u^2 \phi(u) \, du = \Phi(u) - u \phi(u) \]
\[ \int u \phi(u) \, du = -\phi(u) \]
\[ \int \phi(u) \, du = \Phi(u) \]

\[ \Rightarrow \text{applying the three formulas we obtain} \]
\[ E[x^2 | x > \alpha] = \frac{1}{\Phi(\alpha)} \left[ \int \left( \Phi(u) - u \phi(u) - 2 \mu \phi(u) \right) \, du + \int \mu^2 \Phi(u) \, du \right] \]
\[ = \frac{\Gamma (\sigma^2 + \mu^2) (1 - \Phi(\alpha)) + \sigma^2 (\alpha \sigma + 2 \mu) \phi(\alpha)}{\Gamma [1 - \Phi(\alpha)]} \]
\[ = \sigma^2 + \mu^2 + \sigma^2 \alpha \cdot \lambda(\alpha) + 2 \sigma \mu \lambda(\alpha) \]

Hence,
\[ \text{Var}(x^2 | x > \alpha) = E(x^2 | x > \alpha) - E(x | x > \alpha)^2 \]
\[ = \sigma^2 + \mu^2 + \sigma^2 \alpha \lambda(\alpha) + 2 \sigma \mu \lambda(\alpha) \]
\[ - \left[ \mu + \sigma \lambda(\alpha) \right]^2 \]
\[ = \sigma^2 \left[ 1 - \lambda(\alpha)(\lambda(\alpha) - \alpha) \right] \]