14.381 PS # 4 SOLUTIONS
STATISTICS FALL 2004

TA: JOSÉ TESSADA
(tessada@mit.edu)

1. C&B 3.28

(a)

(i) μ known

$$f(x|\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

$$h(x) = 1, \qquad c(\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \mathbb{1}(x \in (0,\infty)),$$

$$w_1(\sigma^2) = -\frac{1}{2\sigma^2}, \qquad t_1(x) = (x-\mu)^2.$$

(ii) σ^2 known

$$f(x|\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\exp\left(\frac{-\mu^2}{\sigma^2}\right)\exp\left(\frac{\mu x}{\sigma^2}\right)$$

$$h(x) = \exp\left(-\frac{x^2}{2\sigma^2}\right), \qquad c(\mu) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\mu^2}{2\sigma^2}\right),$$

$$w_1(\mu) = \mu, \qquad t_1(x) = \frac{x}{\sigma^2}$$

(b) (i) α known

$$f(x|\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} \exp(-x/\beta)$$

$$h(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \cdot \mathbb{1}(x>0), \qquad c(\beta) = \frac{1}{\beta^\alpha},$$

$$w_1(\beta) = 1/\beta, \qquad t_1(x) = -x.$$

(ii) β known

$$f(x|\alpha) = e^{(-x/\beta)} \frac{1}{\Gamma(\alpha)\beta^\alpha} \exp((\alpha-1)\log x)$$

$$h(x) = e^{(-x/\beta)} \cdot \mathbb{1}(x>0), \qquad c(\alpha) = \frac{1}{\Gamma(\alpha)\beta^\alpha},$$

$$w_1(\alpha) = \alpha - 1, \qquad t_1(x) = \log x$$

(iii) α, β unknown

$$f(x|\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha} \cdot \exp\left((\alpha-1)\log x - \frac{x}{\beta}\right)$$

$$h(x) = \mathbb{1}(x>0), \qquad c(\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha},$$

$$w_1(\alpha) = \alpha - 1, \qquad t_1(x) = \log x,$$

$$w_2(\alpha,\beta) = -1/\beta, \qquad t_2(x) = x.$$

(c) (i) α known,

$$h(x) = x^{\alpha-1}\mathbb{1}(x \in [0,1]), \qquad c(\beta) = \frac{1}{B(\alpha,\beta)},$$

$$w_1(\beta) = \beta - 1, \qquad t_1(x) = \log(1-x).$$

(ii) β known

$$h(x) = (1-x)^{\beta-1} \cdot \mathbb{1}(x \in [0,1]), \qquad c(\alpha) = \frac{1}{B(\alpha,\beta)},$$

$$w_1(\alpha) = \alpha - 1, \qquad t_1(x) = \log x$$

(iii) α, β unknown

$$h(x) = \mathbb{1}(x \in [0,1]), \quad C(\alpha,\beta) = \frac{1}{B(\alpha,\beta)},$$

$$w_1(\alpha) = \alpha - 1, \quad t_1(x) = \log x, \quad w_2(\beta) = \beta - 1, \quad t_2(x) = \log(1-x).$$

2. C&B 3.42

(a) Let $\theta_1 > \theta_2$. Let $X_1 \sim f(x-\theta_1)$ and $X_2 \sim f(x-\theta_2)$. Let $F(z)$ be the cdf corresponding to $f(z)$ and let $Z \sim f(z)$. Then

$$\begin{aligned} F(x|\theta_1) &= P(X_1 \le x) = P(Z+\theta_1 \le x) = P(Z \le x-\theta_1) = F(x-\theta_1) \\ &\le F(x-\theta_2) = P(Z \le x-\theta_2) = P(Z+\theta_2 \le x) = P(X_2 \le x) \\ &= F(x|\theta_2) \end{aligned}$$

$$\Rightarrow \quad F(x|\theta_1) \le F(x|\theta_2)$$

The inequality is because $x-\theta_2 > x-\theta_1$, and F is nondecreasing. To get strict inequality for some x, let $(a,b]$ be an interval of length $\theta_1 - \theta_2$ with $P(a < Z \le b) = F(b) - F(a) > 0$. Let $x = a + \theta_1$, then

$$\begin{aligned} F(x/\theta_1) &= F(x-\theta_1) = F(a+\theta_1-\theta_1) = F(a) \\ &< F(b) = F(a+\theta_1-\theta_2) = F(x-\theta_2) = F(x|\theta_2) \end{aligned}$$

(b) Let $\sigma_1 > \sigma_2$. Let $X_1 \sim f(x/\sigma_1)$ and $X_2 \sim f(x/\sigma_2)$. Let $F(z)$ be the cdf corresponding to $f(z)$ and let $Z \sim f(z)$. Then, for $x > 0$,

$$\begin{aligned} F(x|\sigma_1) &= P(X_1 \le x) = P(\sigma_1 Z \le x) = P(Z \le x/\sigma_1) = F(x/\sigma_1) \\ &\le F(x/\sigma_2) = P(Z \le x/\sigma_2) = P(\sigma_2 Z \le x) = P(X_2 \le x) \\ &= F(x|\sigma_2) \end{aligned}$$

$$\Rightarrow \quad F(x|\sigma_1) \le F(x|\sigma_2)$$

The inequality is because $x/\sigma_2 > x/\sigma_1$ (because $x>0$ and $\sigma_1 > \sigma_2 > 0$), and F is nondecreasing.
For $x \le 0$, $F(x|\sigma_1) = P(X_1 \le x) = 0 = P(X_2 \le x) = F(x|\sigma_2)$.
To get strict inequality for some x, let $(a, b]$ be an interval such that $a>0$, $b/a = \sigma_1/\sigma_2$ and $P(a < Z \le b) = F(b) - F(a) > 0$. Let $x = a\sigma_1$. Then,

$$
\begin{aligned}
F(x|\sigma_1) &= F(x/\sigma_1) = F(a\sigma_1/\sigma_1) = F(a) \\
&< F(b) = F(a\sigma_1/\sigma_2) = F(x/\sigma_2) \\
&= F(x|\sigma_2)
\end{aligned}
$$

$$\Rightarrow \quad F(x|\sigma_1) < F(x|\sigma_2).$$

3. C&B 4.1

Since the distribution is uniform, the easiest way to calculate these probabilities is as the ratio of areas, the total area being 4.

(a) the circle $x^2 + y^2 \le 1$ has area π, so $P(X^2 + Y^2 \le 1) = \pi/4$

(b) the area below the line $y = 2x$ is half of the area of the square, so $P(2X - Y > 0) = 1/2$

(c) clearly $P(|X+Y| < 2) = 1$

4. C&B 4.4

(a) $\int_0^1 \int_0^2 c(x+2y)\,dx\,dy = 4c = 1$

$\Rightarrow \quad c = 1/4.$

(b) $f_X(x) = \begin{cases} \int_0^1 \frac{1}{4}(x+2y)\,dy = \frac{1}{4}(x+1) & 0<x<2 \\ 0 & \text{otherwise} \end{cases}$

(c) $F_{X,Y}(x,y) = P(X \le x,\ Y \le y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(v,u)\,du\,dv.$

Notice that the way this integral is calculated depends on the values of x and y. For example, for $0<x<2$ and $0<y<1$,

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dv\,du = \int_0^x \int_0^y \frac{1}{4}(u+2v)\,dv\,du$$

$$= \frac{x^2 y}{8} + \frac{y^2 x}{4}$$

Now, if we take $0<x<2$ and $y \ge 1$

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(u,v)\,dv\,du = \int_0^x \int_0^1 \frac{1}{4}(u+2v)\,dv\,du = \frac{x^2}{8} + \frac{x}{4}$$

The complete definition of $F_{X,Y}(x,y)$ is:

$$F_{X,Y}(x,y) = \begin{cases} 0 & x \le 0 \text{ or } y \le 0 \\ \frac{x^2 y}{8} + \frac{y^2 x}{4} & 0 < x < 2 \text{ and } 0 < y < 1 \\ y/2 + y^2/2 & x \ge 2 \text{ and } 0 < y < 1 \\ \frac{x^2}{8} + \frac{x}{4} & 0 < x < 2 \text{ and } y \ge 1 \\ 1 & 2 \le x \text{ and } 1 \le y \end{cases}$$

(d) The function $z = g(x) = 9/(x+1)^2$ is monotone on $0 < x < 2$, so we use theorem 2.1.5 to obtain
$f_Z(z) = 9/(8z^2)$, $\quad 1 < z < 9$.

5. C&B 4.9

$$P(a \le X \le b,\ c \le Y \le d) = P(X \le b,\ c \le Y \le d) - P(X \le a,\ c \le Y \le d)$$

$$= P(X \le b, Y \le d) - P(X \le b, Y \le c) - P(X \le a, Y \le d) + P(X \le a, Y \le c)$$

$$= F(b,d) - F(b,c) - F(a,d) + F(a,c)$$

$$= F_X(b)F_Y(d) - F_X(b)F_Y(c) - F_X(a)F_Y(d) + F_X(a)F_Y(c)$$

$$= P(X \le b)\left[P(Y \le d) - P(Y \le c)\right] - P(X \le a)\left[P(Y \le d) - P(Y \le c)\right]$$

$$= P(X \le b)\,P(c \le Y \le d) - P(X \le a)\,P(c \le Y \le d)$$

$$= P(a \le X \le b)\,P(c \le Y \le d)$$

6. A, B are independent.

$A \sim U(3,5)$

$B \sim U(3,5)$

$$\Rightarrow \quad f_{A,B}(a,b) = \begin{cases} 1/4 & \text{if } a,b \in (3,5) \\ 0 & \text{otherwise} \end{cases}$$

For $x < 0$, $F_X(x) = P(X \leq x) = 0$

$x \geq 2$, $F_X(x) = P(X \leq x) = 1$

$$F_X(0) = P(X \leq 0) = P(B \leq A) = \int_3^5 \int_3^a \frac{1}{4}\, db\, da = \frac{1}{2}$$

For $0 < x < 2$,

$$F_X(x) = P(X \leq x) = 1 - P(X > x) = 1 - P(B > A + x)$$

$$= 1 - \int_3^{5-x} \int_{a+x}^{5} \frac{1}{4}\, db\, da = 1 - \frac{1}{4} \int_0^{2-x} \int_{a+x}^{2} db\, da$$

$$= 1 - \frac{1}{4} \int_0^{2-x} [(2-x) - a]\, da = 1 - \frac{1}{4} \left[(2-x)a - \frac{a^2}{2} \right]_0^{2-x}$$

$$= 1 - \frac{(2-x)^2}{8}$$

Hence,

$$F_X(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1/2 & \text{if } x = 0 \\ 1 - \frac{(2-x)^2}{8} & \text{if } 0 < x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$