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PS#5 SOLUTIONS

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1.- C&B 4.20

(a) The complicated part comes from the fact that you cannot determine the sign of X_2 from Y_1 and Y_2.

Take two different sets: $-\infty < X_1 < \infty$, $X_2 > 0$
$-\infty < X_1 < \infty$, $X_2 < 0$

The set with $X_2 = 0$ plays the same role as A_0 in the univariate case.

The inverse transformations are:

$$x_1 = y_2\sqrt{y_1}$$
$$x_2 = \sqrt{y_1 - y_1 y_2^2} \qquad x_2 > 0$$

and the Jacobian (see the formula in the book) is

$$J_1 = \begin{vmatrix} \frac{1}{2}\frac{y_2}{\sqrt{y_1}} & \sqrt{y_1} \\ \frac{1}{2}\frac{\sqrt{1-y_2^2}}{\sqrt{y_1}} & \frac{y_2\sqrt{y_1}}{\sqrt{1-y_2^2}} \end{vmatrix} = \frac{1}{2\sqrt{1-y_2^2}}$$

When we take $x_2 < 0$

$$x_1 = y_2\sqrt{y_1} \qquad x_2 = -\sqrt{y_1 - y_1 y_2^2}$$

and $J_2 = -J_1$ by property of the determinants

now we can compute

$$f_{Y_1 Y_2}(y_1, y_2) = 2\left[\frac{1}{2\pi\sigma^2} e^{-y_1/(2\sigma^2)} \frac{1}{2\sqrt{1-y_2^2}}\right]$$

because the formula is the same in both cases.

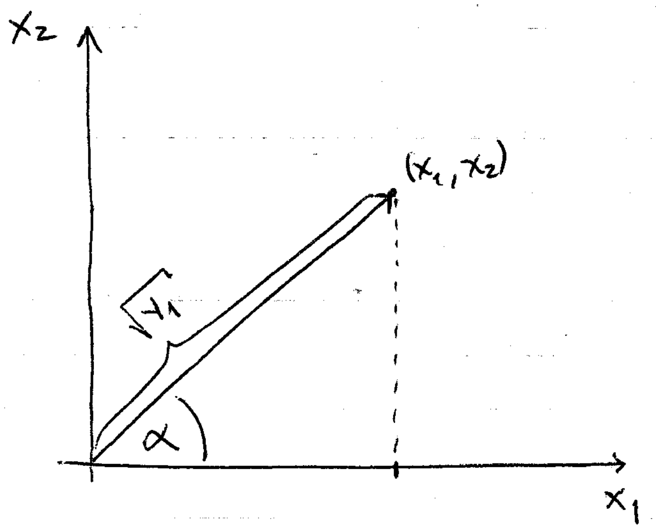
$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi\sigma^2} e^{-y_1/(2\sigma^2)} \frac{1}{\sqrt{1-y_2^2}}$$

$0 < y_1 < \infty$
$-1 < y_2 < 1$

(b) Y_1 and Y_2 are independent because the joint pdf factorizes:

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{\mathbb{1}(y_1 \in (0,\infty))}{2\pi\sigma^2} e^{-y_1/(2\sigma^2)} \cdot \frac{\mathbb{1}(y_2 \in (-1,1))}{\sqrt{1-y_2^2}}$$

$$= h(y_1) \cdot g(y_2)$$



$Y_2 = \text{cosine}(\alpha)$

$\Rightarrow$ the distance from the origin is independent of the orientation (as measured by the angle).

2.-

(a) $E(X) = E\left(E(X|\theta)\right) = E(\theta) = 1/2$

$$Var(X) = E\left(Var(X|\theta)\right) + Var\left(E(X|\theta)\right) = E(\theta^2) + Var(\theta) = 5/12$$

$$Cov(\theta, X) = E(\theta X) - E(\theta)E(X) = E\left(E(\theta X/\theta)\right) - \frac{1}{4}$$

$$= E\left(\theta \cdot E(X|\theta)\right) - \frac{1}{4} = E(\theta^2) - \frac{1}{4} = 1/12$$

(b) Intuitively you should be guessing $\left(\frac{X}{\theta}\right) \sim N(1,1)$. But we need to use a transformation to be sure of that result and to prove independence:

Define $U = X/\theta$ and $V = \theta$ $\Rightarrow$ $X = UV$ and $\theta = V$. Thus,

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ 0 & 1 \end{vmatrix} = v$$

Now, notice that

$$f_{X,Y}(x,y) = \frac{1}{\sqrt{2\pi}\,\theta} \cdot \exp\left\{-\frac{1}{2}\left(\frac{x-\theta}{\theta}\right)^2\right\} \cdot \mathbb{1}(\theta \in (0,1))$$

$\Rightarrow$

$$f_{U,V}(u,v) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}(u-1)^2\right\} \mathbb{1}(v \in (0,1))$$

$\Rightarrow$ U and V are independent and

$\frac{X}{\theta} = U \sim N(1,1)$

$\theta = V \sim U(0,1)$ $\longrightarrow$ something we were sure about!

3.- From theorem 4.2.14 (C&B) we know $U \sim N(\mu+\gamma, 2\sigma^2)$ and $V \sim N(\mu-\gamma, 2\sigma^2)$. It remains to show that they are independent. Proceed as follows:

by independence

$$f_{XY} = f_X f_Y = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[(x-\mu)^2+(y-\gamma)^2\right]\right\}$$

Use the following change of variables: $u = x+y$
$v = x-y$

then $x = \frac{1}{2}(u+v)$, $y = \frac{1}{2}(u-v)$

Using the same formula as in exercise 1 of this problem set:

$$f_{UV}(u,v) = \frac{1}{2\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[\left(\left(\frac{u+v}{2}\right)-\mu\right)^2 + \left(\left(\frac{u-v}{2}\right)-\gamma\right)^2\right]\right\} \cdot \frac{1}{2}$$

$$= \frac{1}{4\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\left[2\left(\frac{u}{2}\right)^2 - u(\mu+\gamma) + \frac{(\mu+\gamma)^2}{2} + 2\left(\frac{v}{2}\right)^2 - v(\mu-\gamma) + \frac{(\mu-\gamma)^2}{2}\right]\right\}$$

$$= \frac{1}{4\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\cdot\frac{1}{2}\left[(u-(\mu+\gamma))^2 + (v-(\mu-\gamma))^2\right]\right\}$$

and it should be clear that we can write it as the product of $g(u)$ and $h(v)$

$$g(u) = \frac{1}{4\pi\sigma^2} \exp\left\{-\frac{1}{2\sigma^2}\cdot\frac{1}{2}(u-(\mu+\gamma))^2\right\}$$

$$h(v) = \exp\left\{-\frac{1}{2\sigma^2}\cdot\frac{1}{2}\cdot(v-(\mu-\gamma))^2\right\}$$

$\Rightarrow$ U and V are independent.

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4.- C&B 4.31

$Y|X \sim \text{Binomial}(n, x)$

$X \sim U(0,1)$

(a) $E(Y) = E\{E(Y|X)\} = E(nX) = \frac{n}{2}$

$$Var(Y) = Var(E(Y|X)) + E(Var(Y|X))$$

$$= Var(nX) + E(nX(1-X)) = \frac{n^2}{12} + \frac{n}{6}$$

(b) $P(Y=y, X \leq x) = \binom{n}{y} x^y (1-x)^{n-y}$, $y = 0, 1, \dots, n$; $0 < x < 1$

(c)

$$P(Y=y) = \binom{n}{y} \frac{\Gamma(y+1)\,\Gamma(n-y+1)}{\Gamma(n+2)}$$

$$= \int_0^1 \binom{n}{y} x^y (1-x)^{n-y}\, dx$$

5.- C&B 4.42

$$\rho_{XY,Y} = \frac{Cov(XY, Y)}{\sigma_{XY}\,\sigma_Y} = \frac{E(XY^2) - \mu_{XY}\mu_Y}{\sigma_{XY}\,\sigma_Y}$$

$$\overset{\text{by independence}}{=} \frac{EX\,EY^2 - \mu_X \mu_Y \mu_Y}{\sigma_{XY}\,\sigma_Y}$$

Need to obtain an expression for σ_{XY}

$$\sigma_{XY}^2 = E[(XY)^2] - [E[XY]]^2 = E(X^2)\,E(Y^2) - E(X)^2\,E(Y)^2$$

$$= (\sigma_X^2 + \mu_X^2)(\sigma_Y^2 + \mu_Y^2) - \mu_X^2 \mu_Y^2 = \sigma_X^2 \sigma_Y^2 + \sigma_X^2 \mu_Y^2 + \sigma_Y^2 \mu_X^2$$

therefore,

$$\rho_{XY,Y} = \frac{\mu_X(\sigma_Y^2+\mu_Y^2) - \mu_X\mu_Y^2}{(\sigma_X^2\sigma_Y^2 + \sigma_X^2\mu_Y^2 + \sigma_Y^2\mu_X^2)^{1/2}\,\sigma_Y}$$

$$= \frac{\mu_X\sigma_Y}{(\sigma_X^2\sigma_Y^2+\sigma_X^2\mu_Y^2+\sigma_Y^2\mu_X^2)^{1/2}}$$

6.- C&B 4.46

(a) $E(X) = a_X \overset{0}{\cancel{E(Z_1)}} + b_X \overset{0}{\cancel{E(Z_2)}} + E(C_X) = C_X$

$Var(X) = a_X^2 \overset{1}{\cancel{Var(Z_1)}} + b_X^2 \overset{1}{\cancel{Var(Z_2)}} + \overset{0}{\cancel{Var(C_X)}} = a_X^2 + b_X^2$

$E(Y) = a_Y \overset{0}{\cancel{E(Z_1)}} + b_Y \overset{0}{\cancel{E(Z_2)}} + E(c_Y) = c_Y$

$Var(\cancel{X}) = a_Y^2 Var(Z_1) + b_Y^2 Var(Z_2) + Var(C_Y)$

$$Cov(X,Y) = E(XY) - E(X)E(Y)$$
$$= E[(a_X a_Y Z_1^2 + b_X b_Y Z_2^2 + \cancel{C_X C_Y} + a_X b_Y \overset{0}{\cancel{Z_1 Z_2}} + a_X C_Y \overset{0}{\cancel{Z_1}}$$
$$+ b_X a_Y \underset{0}{\cancel{Z_1 Z_2}} + b_X C_Y \overset{0}{\cancel{Z_2}} + C_X a_Y \overset{0}{\cancel{Z_1}} + C_X b_Y \overset{0}{\cancel{Z_2}} - \cancel{C_X C_Y}]$$
$$= a_X a_Y + b_X b_Y$$

(b) $E(X) = \mu_X$, $E(Y) = \mu_Y$ are clear!

$V(X) = a_X^2 + b_X^2 = \left(\frac{1+\rho}{2}\right)\sigma_X^2 + \left(\frac{1-\rho}{2}\right)\sigma_X^2 = \sigma_X^2$

$V(Y) = a_Y^2 + b_Y^2 = \left(\frac{1+\rho}{2}\right)\sigma_Y^2 + \left(\frac{1-\rho}{2}\right)\sigma_Y^2 = \sigma_Y^2$

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \left[\left(\frac{1+\rho}{2}\right)\sigma_X\sigma_Y - \left(\frac{1-\rho}{2}\right)\sigma_X\sigma_Y\right] / \sigma_X\sigma_Y = \frac{1+\rho+1-\rho}{2} = \rho$

(c) Solve for Z_1 and Z_2 in terms of X, Y.

Let me define

$$D = a_x b_y - a_y b_x = -\sqrt{1-\rho^2}\,\sigma_x \sigma_y$$

$$Z_1 = \frac{X - b_x Z_2 - c_x}{a_x} \qquad Z_1 = \frac{Y - b_y Z_2 - c_y}{a_y}$$

$$Z_1 = Z_1 \Rightarrow \quad a_y X - a_y b_x Z_2 - a_y c_x = a_x Y - a_x b_y Z_2 - a_x c_y$$

$$Z_2 = \frac{a_x Y - a_y X + (a_y c_x - a_x c_y)}{D}$$

We can write it in terms of $\sigma_x, \sigma_y, \mu_x, \mu_y$ and ρ

$$Z_2 = \frac{\sigma_y(X-\mu_x) - \sigma_x(Y-\mu_y)}{\sqrt{2(1-\rho)}\,\sigma_x \sigma_y}$$

plug into Z_1

$$Z_1 = \frac{X - \sqrt{\frac{1-\rho}{2}}\,\sigma_x \left[\frac{\sigma_y(X-\mu_x) - \sigma_x(Y-\mu_y)}{\sqrt{2(1-\rho)}\,\sigma_x \sigma_y}\right] - \mu_x}{\sqrt{\frac{1+\rho}{2}}\cdot \sigma_x}$$

$$= \frac{X - \frac{1}{2}(X-\mu_x) + \frac{1}{2}\left(\frac{\sigma_x}{\sigma_y}\right)(Y-\mu_y) - \mu_x}{\sqrt{\frac{1+\rho}{2}}\cdot\sigma_x}$$

$$= \left[\frac{1}{2}(X-\mu_x) + \frac{1}{2}\left(\frac{\sigma_x}{\sigma_y}\right)(Y-\mu_y)\right] \Big/ \sqrt{\frac{1+\rho}{2}}\cdot \sigma_x$$

$$= \left[\sigma_y(X-\mu_x) + \sigma_x(Y-\mu_y)\right] \Big/ \sqrt{2(1+\rho)}\,\sigma_x\sigma_y$$

$$Z_1 = \sigma_y(X-\mu_x) + \sigma_x(Y-\mu_y) \Big/ \sqrt{2(1+\rho)}\,\sigma_x\sigma_y$$

$$Z_2 = \sigma_y(X-\mu_x) - \sigma_x(Y-\mu_y) \Big/ \sqrt{2(1+\rho)}\,\sigma_x\sigma_y$$

The Jacobian of the transformation is

$$J = \begin{vmatrix} \partial z_1/\partial x & \partial z_1/\partial y \\ \partial z_2/\partial x & \partial z_2/\partial y \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2(1+\rho)}\,\sigma_x} & \frac{1}{\sqrt{2(1+\rho)}\,\sigma_y} \\ \frac{1}{\sqrt{2(1+\rho)}\,\sigma_x} & \frac{-1}{\sqrt{2(1-\rho)}\,\sigma_y} \end{vmatrix}$$

$$= -\frac{1}{2\sqrt{1-\rho^2}\,\sigma_x\sigma_y} - \frac{1}{2\sqrt{1-\rho^2}\,\sigma_x\sigma_y} = \frac{1}{D}$$

$$|J| = \frac{1}{\sqrt{1-\rho^2}\,\sigma_x\sigma_y}$$

So, now we can write

$$f_{x,y}(x,y) = \left(\frac{1}{\sqrt{2\pi}} e^{\left\{-\frac{1}{2}\left[\sigma_y(x-\mu_x)+\sigma_x(y-\mu_y)\right]^2 / 2(1+\rho)\sigma_x^2\sigma_y^2\right\}}\right)$$

$$\cdot \left(\frac{1}{\sqrt{2\pi}} e^{\left\{-\frac{1}{2}\left[\sigma_y(x-\mu_x)-\sigma_x(y-\mu_y)\right]^2 / 2(1-\rho)\sigma_x^2\sigma_y^2\right\}}\right) \cdot \frac{1}{\sqrt{1-\rho^2}\,\sigma_x\sigma_y}$$

$$= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)} \cdot \left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

$$-\infty < x < \infty, \quad -\infty < y < \infty$$

which is a bivariate normal pdf.

(d) Another solution is:

$$a_x = \rho\sigma_x \quad b_x = \sqrt{1-\rho^2}\,\sigma_x$$
$$a_y = \sigma_y \quad b_y = 0$$
$$c_x = \mu_x$$
$$c_y = \mu_y$$

This comes from the fact that we do not have enough equations to solve the system.