1. Let $X_i$ = weight of the $i$th booklet in package. The $X_i$s are iid with $E(X_i) = 1$ and $\text{Var}(X_i) = 0.05^2$. We want to approximate $\Pr\left(\sum_{i=1}^{100} X_i > 100.4\right) = \Pr\left(\sum_{i=1}^{100} \frac{X_i}{100} > 1.004\right)$. By the CLT, $\Pr(\bar{X} > 1.004) \approx \Pr\left(Z > \frac{(1.004 - 1)}{0.05\sqrt{100}}\right) = \Pr(Z > 0.8) = 0.2119$.

The only assumption we need is the weight of the different booklets being independent. The first two moments exist, so we can apply a CLT for independent and identically distributed (iid) data.

2. (a) For any $\varepsilon > 0$,

$$\Pr\left(\sqrt{n} \frac{X_n - \mu}{\sigma} > \varepsilon\right) = \Pr\left(\frac{\sqrt{n} X_n - \sqrt{n} \mu}{\sqrt{n} \sigma} > \varepsilon\right) = \Pr\left(\frac{|X_n - \mu|}{\sigma} > \varepsilon\right) \leq \Pr\left(|X_n - \mu| > \varepsilon\sigma\right) \to 0,$$

as $n \to \infty$, since $X_n \xrightarrow{p} \mu$. Thus, $\sqrt{n} X_n \xrightarrow{p} \mu$.

(b) For any $\varepsilon > 0$,

$$\Pr\left(\left|\frac{X_n}{\sigma} - 1\right| \leq \varepsilon\right) = \Pr\left(1 + \varepsilon \leq \frac{X_n}{\sigma} \leq 1 - \varepsilon\right) = \Pr\left(a - \frac{a \varepsilon}{1 + \varepsilon} \leq X_n \leq a + \frac{a \varepsilon}{1 - \varepsilon}\right) \geq \Pr\left(a - \frac{a \varepsilon}{1 + \varepsilon} \leq X_n \leq a + \frac{a \varepsilon}{1 + \varepsilon}\right) \to 1,$$

as $n \to \infty$, since $X_n \xrightarrow{p} \mu$. Thus $\frac{X_n}{\sigma} \xrightarrow{p} 1$.

(c) $S^2 = \frac{\sigma^2}{n}$. By (a), $S_n = \sqrt{\frac{S^2}{n}} \xrightarrow{p} \sigma$. By (b), $\frac{S_n}{\sigma} \xrightarrow{p} 1$.

3. Find the method of moments estimator of the following parameters:

(a) $\lambda$ for the case of an exponential distribution: $f(x|\theta) = \frac{1}{\lambda} \exp(-x/\lambda), \quad 0 < x < \infty, \lambda > 0$;

We know that if $X \sim \text{Exponential}(\lambda)$, then $E(X) = \lambda$.

The methods of moments estimator is

$$\hat{\lambda}_{MM} = \bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i.$$

(b) $\sigma$ for $N(\mu, \sigma^2)$ when $\mu$ is known;

In the case of the Normal distribution the first moment $E(X)$ is not giving us any information as we know the value of $\mu$. We can use higher order moments, in particular, $E\left(X^2\right) = \sigma^2 + \mu^2$, so we have a methods of moments estimator using the second order moments:

$$\hat{\sigma}_{MM} = \begin{cases} \sqrt{\frac{1}{n} \left(\sum_{i=1}^{n} X_i^2\right) - \mu^2} & \text{if } \frac{1}{n} \left(\sum_{i=1}^{n} X_i^2\right) \geq \mu^2 \\ \text{not defined} & \text{if } \frac{1}{n} \left(\sum_{i=1}^{n} X_i^2\right) < \mu^2 \end{cases}.$$

If you happen to be in the second case you can try higher order moments.
(c) $\theta$ if the pdf is $f(x|\theta) = \theta x^{-2}, \quad 0 < \theta \leq x < \infty$.

We cannot find a method of moments estimator using any moment like $E(X^k), \ k \geq 1$. To see this note the following

\[
E(X^k) = \int_0^\infty x^k \theta x^{-2} \, dx = \int_0^\infty x^{k-2} \theta \, dx \geq \int_0^\infty \theta^{k-1} \, dx
\]

and the last integral does not exist for $k \geq 1$. However, it is easy to see that $E(X^{-1})$ exists

\[
E(X^{-1}) = \int_0^\infty x^{-1} \theta x^{-2} \, dx = \int_0^\infty x^{-3} \theta \, dx = -\frac{\theta}{2} x^{-2} \bigg|_0^\infty = \frac{1}{2\theta}.
\]

Thus, we can define a method of moments estimator for $\theta$ as

\[
\hat{\theta}_{MM} = \frac{n}{2} \frac{1}{\sum_{i=1}^n \frac{1}{X_i}}.
\]