14.381 Solutions Problem Set 7
Statistics Fall, 2004

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1. Let X_i = weight of the ith booklet in package. The X_is are iid with $E(X_i) = 1$ and $Var(X_i) = 0.05^2$. We want to approximate $\Pr\left(\sum_{i=1}^{100} X_i > 100.4\right) = \Pr\left(\sum_{i=1}^{100} \frac{X_i}{100} > 1.004\right) = \Pr\left(\overline{X} > 1.004\right)$.

   By the CLT, $\Pr\left(\overline{X} > 1.004\right) \approx \Pr\left(Z > \frac{(1.004-1)}{(0.05/10)}\right) = \Pr(Z > 0.8) = 0.2119.$

   The only assumption we need is the weight of the different booklets being independent. The first two moments exist, so we can apply a CLT for idependent and identically distributed (iid) data.

2.

   (a) For any $\varepsilon > 0$,

$$\begin{aligned}\Pr\left(|\sqrt{X_n} - \sqrt{a}| > \varepsilon\right) &= \Pr\left(|\sqrt{X_n} - \sqrt{a}|\ |\sqrt{X_n} + \sqrt{a}| > \varepsilon|\sqrt{X_n} + \sqrt{a}|\right) \\ &= \Pr\left(|X_n - a| > \varepsilon|\sqrt{X_n} + \sqrt{a}|\right) \\ &\leq \Pr\left(|X_n - a| > \varepsilon\sqrt{a}\right) \to 0,\end{aligned}$$

   as $n \to \infty$, since $X_n \xrightarrow{p} a$. Thus, $\sqrt{X_n} \xrightarrow{p} \sqrt{a}$.

   (b) For any $\varepsilon > 0$,

$$\begin{aligned}\Pr\left(\left|\frac{a}{X_n} - 1\right| \leq \varepsilon\right) &= Pr\left(\frac{a}{1+\varepsilon} \leq X_n \leq \frac{a}{1-\varepsilon}\right) \\ &= \Pr\left(a - \frac{a\varepsilon}{1+\varepsilon} \leq X_n \leq a + \frac{a\varepsilon}{1-\varepsilon}\right) \\ &\geq \Pr\left(a - \frac{a\varepsilon}{1+\varepsilon} \leq X_n \leq a + \frac{a\varepsilon}{1+\varepsilon}\right) \qquad \text{because} \quad \left(a + \frac{a\varepsilon}{1+\varepsilon} < a + \frac{a\varepsilon}{1-\varepsilon}\right) \\ &= \Pr\left(|X_n - a| \leq \frac{a\varepsilon}{1+\varepsilon}\right) \to 1\end{aligned}$$

   as $n \to \infty$, since $X_n \xrightarrow{p} a$. Thus $\frac{a}{X_n} \xrightarrow{p} 1$.

   (c) $S^2 \xrightarrow{p} \sigma^2$. By (a), $S_n = \sqrt{S_n^2} \xrightarrow{p} \sqrt{\sigma^2} = \sigma$. By (b), $\frac{\sigma}{S_n} \xrightarrow{p} 1$.

3. Find the method of moments estimator of the following parameters:

   (a) λ for the case of an exponential distribution: $f(x|\theta) = \frac{1}{\lambda}\exp(-x/\lambda)$, $\quad 0 \leq x < \infty$, $\lambda > 0$;
   We know that if $X \sim Exponential(\lambda)$, then $E(X) = \lambda$.
   The methods of moments estimator is

$$\widehat{\lambda}_{MM} = \overline{X} = \sum_{i=1}^{n} \frac{X_i}{n}.$$

   (b) σ for $N(\mu, \sigma^2)$ when μ is known;
   In the case of the Normal distribution the first moment $E(X)$ is not giving us any information as we know the value of μ. We can use higher order moments, in particular, $E(X^2) = \sigma^2 + \mu^2$, so we have a methods of moments estimator using the second order moments:

$$\widehat{\sigma}_{MM} = \begin{cases} \sqrt{\frac{1}{n}\left(\sum_{i=1}^{n} X_i^2\right) - \mu^2} & \text{if } \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2\right) \geq \mu^2 \\ \text{not defined} & \text{if } \frac{1}{n}\left(\sum_{i=1}^{n} X_i^2\right) < \mu^2 \end{cases}.$$

   If you happen to be in the second case you can try higher order moments.

(c) θ if the pdf is $f(x|\theta) = \theta x^{-2}, \qquad 0 < \theta \leq x < \infty.$

We cannot find a method of moments estimator using any moment like $E\left(X^k\right),\ k \geq 1$. To see this note the following

$$
\begin{aligned}
E\left(X^k\right) &= \int_\theta^\infty x^k \theta x^{-2} dx \\
&= \int_\theta^\infty x^{k-2} \theta dx \\
&\geq \int_\theta^\infty \theta^{k-1} dx
\end{aligned}
$$

and the last integral does not exist for $k \geq 1$. However, it is easy to see that $E\left(X^{-1}\right)$ exists

$$
\begin{aligned}
E\left(X^k\right) &= \int_\theta^\infty x^{-1} \theta x^{-2} dx \\
&= \int_\theta^\infty x^{-3} \theta dx \\
&= -\frac{\theta}{2} x^{-2} \Big|_\theta^\infty = \frac{1}{2\theta}.
\end{aligned}
$$

Thus, we can define a method of moments estimator for θ as

$$
\widehat{\theta}_{MM} = \frac{n}{2} \frac{1}{\sum_{i=1}^n \frac{1}{X_i}}.
$$