14.381 Waiver Examination
Fall 2003

Instructions: This is a closed book exam. You have 90 minutes to answer the questions. Good luck!

1. Given N, let $X_1, ..., X_N$ be an *iid* sample of normal random variables with mean μ and variance σ^2. Let N, the sample size, be a Poisson random variable with

$$P(N = n) = \frac{e^{-\lambda}\lambda^n}{n!} \quad \text{for } n = 0, 1, 2, ...$$

and $\lambda > 0$. Consider the sum of the observations

$$S = \left\{ \begin{array}{cc} \sum_{i=1}^{N} X_i & N > 0 \\ 0 & N = 0 \end{array} \right\}$$

Find ES and ES^2 (Hint: you can use the fact that $EN = \lambda$ and $\text{var}(N) = \lambda$ without proof.)

2. Let $X_1, ..., X_n$ be an *iid* sample from an exponential distribution with mean one. Let $S = n^{-1} \sum_{i=1}^{N} X_i$.

   a) Find the first four moments of S.
   b) What is the limit of ES^3 and ES^4 as $N \to \infty$?
   c) Show that $S \xrightarrow{p} \theta$ as $n \to \infty$ for some θ. What is the value of θ?
   d) Find the limit distribution of $\sqrt{n}(S - \theta)$ as $n \to \infty$.

3. Let $X_1, ..., X_n$ be an *iid* sample from a Binomial distribution with parameter $p > 0$, i.e.

$$P(X = x) = \left(\begin{array}{c} r \\ x \end{array} \right) p^x (1-p)^{r-x} \quad \text{for } x = 0, 1, ..., r$$

Assume that r is known.

   (a) Find the likelihood function for the parameter p.
   (b) Find the maximum likelihood (ML) estimator $\hat{p}$ of p. Is this estimator unbiased?
   (c) Calculate the Fisher information of the sample.

4. v is exponentially distributed with mean one. Conditional on v, t is exponentially distributed with mean $\frac{1}{v}$. Derive the unconditional distribution of t_n. (Hint: use a duration model.)