14.384 Problem Set 1 Solutions
Fall, 2004
Professor: Michael Jansson
T.A.: Pierre Yared

1 ARCH(1) stochastic process

In conventional econometric models, the variance of the disturbance term is assumed to be constant. However many economic time series exhibit periods of unusually large volatility followed by periods of relative tranquility. In such circumstances, the assumption of a constant variance is inappropriate.

Consider the following:

$$\begin{aligned} \epsilon_t &= u_t h_t^{1/2} \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \\ u_t &\sim N(0,1), \text{independent of } \epsilon_{t-1} \end{aligned}$$

(a) Compute the unconditional expectation, the unconditional variance and the autocovariances of ϵ_t. For which values of α_0 and α_1, is ϵ_t white noise ? Now compute the conditional mean of ϵ_t ($E(\epsilon_t \mid \epsilon_{t-1}, \epsilon_{t-2}, ...)$) and its conditional variance. Do we have to restrict further the values of α_0 and α_1 ?

Here is how we compute the unconditional expectation of ϵ_t:

$$E(\epsilon_t) = E\left(u_t h_t^{1/2}\right) = E\left(u_t \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = E(u_t) E\left(\sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = 0$$

where we have used the fact that $u_t \perp \epsilon_{t-1}$ which is assumed in the problem. Here is how we compute the unconditional variance of ϵ_t:

$$\begin{aligned} Var(\epsilon_t) &= E\left(\epsilon_t^2\right) = E\left(u_t^2 h_t\right) = E\left(u_t^2\right) E\left(\alpha_0 + \alpha_1 \epsilon_{t-1}^2\right) = \alpha_0 + \alpha_1 E\left(\epsilon_{t-1}^2\right) = \\ \alpha_0 + \alpha_1 E\left(\epsilon_t^2\right) &= \alpha_0 + \alpha_1 Var(\epsilon_t), \text{ which yields:} \\ Var(\epsilon_t) &= \frac{\alpha_0}{1-\alpha_1} \end{aligned}$$

where we have used the fact that $u_t \perp \epsilon_{t-1}$ and also we have worked under the assumption that ϵ_t is itself stationary (we will need to make further parameter restrictions to ensure that this assumption holds. Now we compute the autocovairances:

$$Cov(\epsilon_t, \epsilon_{t-k}) = E(\epsilon_t \epsilon_{t-k}) = E(u_t h_t u_{t-k} h_{t-k}) = E(u_t) E(h_t u_{t-k} h_{t-k}) = 0$$

ϵ_t is a white noise process if $0 < E\left(\epsilon_t^2\right) < \infty$ which is ensured by $((\alpha_0 > 0) \cap (\alpha_1 < 1)) \cup ((\alpha_0 < 0) \cap (\alpha_1 > 1))$. Next we examine the conditional mean and the conditional variance. Solving for the conditional mean we let $E_{t-1}(\epsilon_t) = E(\epsilon_t | \epsilon_{t-1}, ...)$.

$$E_{t-1}(\epsilon_t) = E\left(u_t h_t^{1/2}\right) = E_{t-1}\left(u_t \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = E_{t-1}(u_t) E_{t-1}\left(\sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}\right) = 0$$

Solving for the conditional variance we achieve:

$$Var_{t-1}(\epsilon_t) = E_{t-1}\left(\epsilon_t^2\right) = E_{t-1}\left(u_t^2 h_t\right) = E_{t-1}\left(u_t^2\right) E_{t-1}\left(\alpha_0 + \alpha_1 \epsilon_{t-1}^2\right) = \alpha_0 + \alpha_1 E_{t-1}\left(\epsilon_{t-1}^2\right) = \alpha_0 + \alpha_1 \epsilon_{t-1}^2$$

ϵ_t is a white noise process if $0 < E_{t-1}\left(\epsilon_t^2\right) < \infty$. Since $\epsilon_{t-1}^2 \in [0, \infty)$, this must be true for all ϵ_{t-1}^2, which implies that $\alpha_0 > 0$ (so that $E_{t-1}\left(\epsilon_t^2\right) > 0$ when $\epsilon_{t-1}^2 = 0$) and $\alpha_1 > 0$ (so that $E_{t-1}\left(\epsilon_t^2\right) > 0$ when $\epsilon_{t-1}^2 \to \infty$). Together with out previous restrictions, we can therefore ensure that ϵ_t is a white noise process by assuming that $(\alpha_0 > 0) \cap (0 < \alpha_1 < 1)$.

(b) Assume you are interested in the following stationary ARMA model:

$$y_t = a_0 + a_1 y_{t-1} + \epsilon_t, \text{ where } \epsilon_t \text{ is defined as above}$$

Compute the unconditional mean and variance of y_t.

First, note that for y_t to be stationary that it is required that $|a_1| < 1$. We solve for the unconditional mean, assuming stationarity:

$$\begin{aligned} E(y_t) &= a_0 + a_1 E(y_{t-1}) + E(\epsilon_t) = a_0 + a_1 E(y_{t-1}), \text{ and since } E(y_t) = E(y_{t-1}) \\ E(y_t) &= \frac{a_0}{1 - a_1} \end{aligned}$$

We know from standard AR(1) models that the unconditional variance will therefore be:

$$Var(y_t) = \frac{Var(\epsilon_t)}{1 - a_1^2} = \frac{1}{1 - a_1^2} \frac{\alpha_0}{1 - \alpha_1}$$

(c) Now consider

$$x_t = \epsilon_t^2 - E(\epsilon_t^2)$$

Is x_t covariance stationary? If so, compute the autocovariance function.

Note that I can rewrite the process for x_t in the following manner:

$$x_t = \epsilon_t^2 - E\left(\epsilon_t^2\right) = \epsilon_t^2 - \frac{\alpha_0}{1-\alpha_1}$$

Furthermore, I can rewrite the process for h_t, adding ϵ_t^2 to both sides as:

$$\epsilon_t^2 + h_t = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \epsilon_t^2 \implies \epsilon_t^2 = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + \left(\epsilon_t^2 - h_t\right) = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + v_t$$

where $v_t = \epsilon_t^2 - h_t$. Plugging back into the equation for x_t, we achieve

$$\begin{aligned} x_t &= \epsilon_t^2 - \frac{\alpha_0}{1-\alpha_1} = \alpha_0 + \alpha_1\epsilon_{t-1}^2 + v_t - \frac{\alpha_0}{1-\alpha_1} = \alpha_1\left(\epsilon_{t-1}^2 - \frac{\alpha_0}{1-\alpha_1}\right) + v_t \\ &= \alpha_1 x_{t-1} + v_t \end{aligned}$$

If I can show that $v_t \sim WN\left(0, \sigma_v^2\right)$ so that $0 < \sigma_v^2 < \infty$, then given $0 < \alpha_1 < 1$, we can use standard formulas for the autocovariance functions of AR(1) processes to determine the autocovariance function for x_t. Therefore, we determine under which circumstances we can think of v_t as being a white noise process. We note that $v_t = \epsilon_t^2 - h_t = u_t^2 h_t - h_t = \left(u_t^2 - 1\right)h_t$, therefore, looking at the unconditional expectation of v_t

$$E\left(v_t\right) = E\left(\left(u_t^2 - 1\right)h_t\right) = E\left(u_t^2 - 1\right)E\left(h_t\right) = 0 \text{ since } u_t \perp \epsilon_{t-1}$$

Now we examine the unconditional variance of v_t.

$$\sigma_v^2 = Var\left(v_t\right) = E\left(v_t^2\right) = E\left(\left(u_t^2 - 1\right)^2 h_t^2\right) = E\left(u_t^2 - 1\right)^2 E\left(h_t^2\right)$$

Note that $E\left(u_t^2 - 1\right)^2 = Var\left(\chi_1\right) = 2$ where χ_1 is a chi-squared distribution, since $u_t \sim N(0,1)$, which means that $u_t^2 \sim \chi_1$. Since $E\left(\chi_1\right) = 1$ by definition, then $E\left(u_t^2\right) = 1$ and $E\left(u_t^2 - 1\right)^2 = E\left(u_t^2 - E\left(u_t^2\right)\right)^2 = Var\left(u_t^2\right) = Var\left(\chi_1\right) = 2$[1]. We now examine the second component $E\left(h_t^2\right)$ to show when it is well behaved:

$$E\left(h_t^2\right) = E\left(\alpha_0 + \alpha_1\epsilon_{t-1}^2\right)^2 = \alpha_0^2 + 2\alpha_1\alpha_0 E\left(\epsilon_{t-1}^2\right) + \alpha_1^2 E\left(\epsilon_{t-1}^4\right) = \alpha_0^2 + 2\alpha_1\alpha_0\frac{\alpha_0}{1-\alpha_1} + \alpha_1^2 E\left(\epsilon_{t-1}^4\right)$$

Note that

$$\begin{aligned} E\left(\epsilon_{t-1}^4\right) &= E\left(u_{t-1}^4 h_{t-1}^2\right) = E\left(u_{t-1}^4\right)E\left(h_{t-1}^2\right) = E\left(u_t^4\right)E\left(h_t^2\right) \\ &= E\left(Var\left(u_t^2\right) + \left(E\left(u_t^2\right)\right)^2\right)E\left(h_t^2\right) = 3E\left(h_t^2\right) \end{aligned}$$

Therefore,

$$E\left(h_t^2\right) = \alpha_0^2 + 2\alpha_1\alpha_0\frac{\alpha_0}{1-\alpha_1} + 3\alpha_1^2 E\left(h_t^2\right)$$

[1] You can find description of Chi-Squared distributions and their moments in any stats book.

which yields:

$$E\left(h_t^2\right) = \alpha_0^2 \frac{\left(1 + 2\frac{\alpha_1}{1-\alpha_1}\right)}{1 - 3\alpha_1^2}$$

This is going to be positive if we impose the additional restriction that $1-3\alpha_1^2 > 0$ so that $|\alpha_1| < 1/\sqrt{3}$. Combining all of our assumptions, therefore $\sigma_v^2 > 0$ if $\alpha_0 > 0$ and $1/\sqrt{3} > \alpha_1 > 0$. Our autocovariances for v_t can be be computed easily:

$$\begin{aligned} Cov\left(v_t, v_{t-k}\right) &= E\left(\left(u_t^2 - 1\right) h_t \left(u_{t-k}^2 - 1\right) h_{t-k}\right) = E\left(u_t^2 - 1\right) E\left[h_t \left(u_{t-k}^2 - 1\right) h_{t-k}\right] \\ &= E\left(u_t^2 - 1\right) E\left[h_t E\left[\left(u_{t-k}^2 - 1\right) h_{t-k}\right]\right] = 0 \end{aligned}$$

Now that we have shown that x_t is covariance stationary under a particular set of circumstances, we can compute its autocovariance function. Since x_t is AR(1), this is very easy

$$\begin{aligned} E\left(x_t\right) &= 0 \\ Var(x_t) &= E(x_t^2) = \frac{1}{1-\alpha_1^2}\sigma_v^2 \\ Cov(x_t, x_{t-k}) &= E(x_t x_{t-k}) = \frac{\alpha_1^k}{1-\alpha_1^2}\sigma_v^2 \end{aligned}$$

2 ARMA(2,2)

Consider the following ARMA(2,2) model

$$\begin{aligned} x_t &= 1.3x_{t-1} - 0.4x_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2} \\ & \epsilon_t \text{ iid N(0,1)} \end{aligned}$$

(a) Is x_t weakly stationary? If so compute the autocovariance function.

We can write this process using lag operators:

$$\begin{aligned} \Phi_2\left(L\right) x_t &= \Theta_2\left(L\right) \epsilon_t \\ \text{where } \Phi_2\left(L\right) &= 1 - 1.3L - 0.4L^2 \\ \text{and } \Theta_2\left(L\right) &= 1 - 1.2L + 0.2L^2 \end{aligned}$$

This is weakly stationary if the roots of the AR part lie outside the unit circle. If we set $1 - 1.3z - 0.4z^2$ and solve for z we achieve $z_1 = 2$ and $z_2 = 1.25$ via the quadratic formula. Since $|z_1| > 1$ and $|z_2| > 1$ the process is stationary.

Let's solve for the autocovariance function (you did not have to do this formally in the problem set): First, it can be shown that assuming stationarity,

$$E\left(x_t\right) = E\left[1.3x_{t-1} - 0.4x_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right] = 0.9E\left[x_t\right] \text{ which implies } E\left(x_t\right) = 0,$$

therefore,

$$\begin{aligned}\gamma_0 &= Cov\left(x_t.x_t\right) = E\left(x_t^2\right) = E\left[x_t\left(1.3x_{t-1} - 0.4x_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] = \\ &\quad 1.3Cov\left(x_t.x_{t-1}\right) - 0.4Cov\left(x_t.x_{t-2}\right) + E\left(x_t\epsilon_t\right) - 1.2E\left(x_t\epsilon_{t-1}\right) + 0.2E\left(x_t\epsilon_{t-2}\right)\end{aligned}$$

Note:

$$\begin{aligned}\gamma_1 &= Cov\left(x_t.x_{t-1}\right) \text{ by definition} \\ \gamma_2 &= Cov\left(x_t.x_{t-2}\right) \text{ by definition} \\ E\left(x_t\epsilon_t\right) &= E\left(\epsilon_t^2\right) = 1 \\ E\left(x_t\epsilon_{t-1}\right) &= 1.3E\left(x_{t-1}\epsilon_{t-1}\right) - 1.2E\left(\epsilon_{t-1}^2\right) = 1.3\left(1\right) - 1.2\left(1\right) = 0.1 \\ E\left(x_t\epsilon_{t-2}\right) &= 1.3E\left(x_{t-1}\epsilon_{t-2}\right) - 0.4E\left(\epsilon_{t-2}x_{t-2}\right) + 0.2E\left(\epsilon_{t-2}^2\right) = 1.3\left(0.1\right) - 0.4\left(1\right) + 0.2\left(1\right) = -0.007\end{aligned}$$

Therefore,

$$\gamma_0 = Cov\left(x_t.x_t\right) = 1.3\gamma_1 - 0.4\gamma_2 + 0.866$$

We can use similar arguments to show that:

$$\begin{aligned}\gamma_1 &= Cov\left(x_t, x_{t-1}\right) = E\left(x_t x_{t-1}\right) = E\left[x_{t-1}\left(1.3x_{t-1} - 0.4x_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] \\ &= 1.3\gamma_0 - 0.4\gamma_1 - 1.2(1) + 0.2(0.1) \\ \gamma_2 &= Cov\left(x_t, x_{t-2}\right) = E\left(x_t x_{t-2}\right) = E\left[x_{t-2}\left(1.3x_{t-1} - 0.4x_{t-2} + \epsilon_t - 1.2\epsilon_{t-1} + 0.2\epsilon_{t-2}\right)\right] \\ &= 1.3\gamma_1 - 0.4\gamma_0 + 0.2(1) \\ \gamma_h &= 1.3\gamma_{h-1} - 0.4\gamma_{h-2} \text{ for } h > 2\end{aligned}$$

We can solve for γ_0,γ_1, and γ_2 explicitly, since we have three equations and three unknowns:

$$\gamma_0 = 10/9, \gamma_1 = 17/90, \text{ and } \gamma_2 = 1/900$$

Therefore, we have a second order difference equation for γ_h with three initial conditions. Note that $\gamma_h = 1.3\gamma_{h-1} - 0.4\gamma_{h-2}$ looks just like the AR component of our equation. The general solution to this second order different equation is therefore

$$\gamma_h = c_1 0.5^h - c_2 0.8^h, \text{where } c_1 \text{and } c_2 \text{ are constants to be determined by initial conditions}$$

Note that 0.5 and 0.8 are the eigenvalues of the system defined by $\gamma_h = 1.3\gamma_{h-1} - 0.4\gamma_{h-2}$ which is why they are the roots which define the motion of γ_h.[2] The final step is to determine c_1 and c_2. Looking at γ_3 and γ_4 yields:

$$\begin{aligned}\gamma_3 &= 1.3\gamma_2 - 0.4\gamma_1 = c_1 0.5^3 - c_2 0.8^3 \\ \gamma_4 &= 1.3\gamma_3 - 0.4\gamma_2 = c_1 0.5^4 - c_2 0.8^4\end{aligned}$$

[2] See Simon and Blume for a review of difference equations and their solutions

Plugging in our values for γ_2 and γ_1 we can solve for c_1 and c_2, so that $c_1 = 1$ and $c_2 = -7/18$. Therefore, the autocovariance function is:

$$\gamma_h = \begin{cases} 10/9 & \text{if } h = 0 \\ 17/90 & \text{if } h = 1 \\ 1/900 & \text{if } h = 2 \\ \left(\frac{1}{2}\right) 0.5^h - \left(\frac{7}{18}\right) 0.8^h & \text{if } h > 2 \end{cases}$$

(b) Is x_t invertible ? If so, find the infinite order MA representation.

The first part of this question has a two part answer. The AR component of the process is invertible and can be represented as an infinite MA process, however, the MA part of the process is not invertible into an AR process since $\Theta_2(L)$ yields quadratic equation $1-1.2z+0.2z^2 = 0$ with roots $z_1 = 5$ and $z_2 = 1$ and z_2 does not lie strictly outside of the unit circle. Since the question asks for an infinite order MA representation and the AR component is invertible, we're all set. The easiest way to show the MA representation is to follow what we did in section with the AR(2) process and define a new error term $\widetilde{\epsilon} = \Theta_2(L)\epsilon_t$.[3] I will take a slightly different approach which will give us a similar answer. Note that I can write an equation for x_t:

$$\begin{aligned} x_t &= \Phi_2^{-1}(L)\Theta_2(L)\epsilon_t = \Psi_\infty(L)\epsilon_t \\ \text{where } \Psi_\infty(L) &= \Psi_0 + \Psi_1 L + \Psi_2 L^2 + ... \end{aligned}$$

We would like to determine Ψ_h. Note that the above equation implies that

$$\left(\Psi_0 + \Psi_1 L + \Psi_2 L^2 + ...\right)\left(1 - 1.3L - 0.4L^2\right) = 1 - 1.2L + 0.2L^2$$

By matching terms within the polynomial for L, we can see that:

$$\begin{aligned} 1 &= \Psi_0 \\ -1.2 &= -1.3\Psi_0 + \Psi_1 \implies \Psi_1 = 0.1 \\ 0.2 &= 0.4\Psi_0 - 1.3\Psi_1 + \Psi_2 \implies \Psi_2 = -0.007 \\ \Psi_h &= 1.3\Psi_{h-1} - 0.4\Psi_{h-2} \text{ for } h > 2 \end{aligned}$$

Note that we already solved this second order difference equation (although we did it with different initial conditions). Therefore, we know that the solution for $h > 2$ is going to be

$$\Psi_h = c_1 0.5^h - c_2 0.8^h, \text{where } c_1 \text{and } c_2 \text{ are constants to be determined by initial conditions}$$

Using initial conditions in the same way that we did before, we find that $c_1 = 1$ and $c_2 = -1/2$. It turns out that these weights will also determine Ψ_h when $h = 1$ and $h = 2$ as well. Therefore, we can characterize Ψ_h in the infinite MA representation:

$$\Psi_h = \begin{cases} 1 & \text{when } h = 0 \\ \left(\frac{1}{2}\right)^h - \frac{1}{2}\left(\frac{4}{5}\right)^h & \text{when } h > 0 \end{cases}$$

[3] For those of you who did not attend section, this is done according to the example in Hamilton p.13